ME 598: Introduction to Robotics

Lecture 2: Forward Kinematics
Denavit-Hartenberg Parameters
Inverse Kinematics

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Date:
By:

Slides adapted from Dr. David J. Cappelleri
Some slides courtesy of Jonathan Fiene, University of Pennsylvania

Review: Topics Covered

manipulator elements

joint types

serial kinematic chains

configuration space

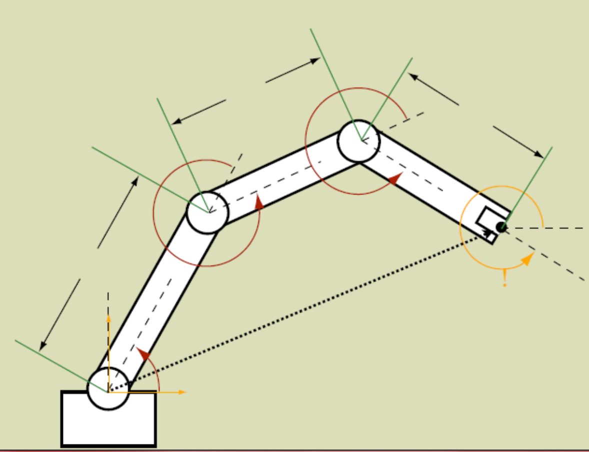
degrees of freedom

workspace

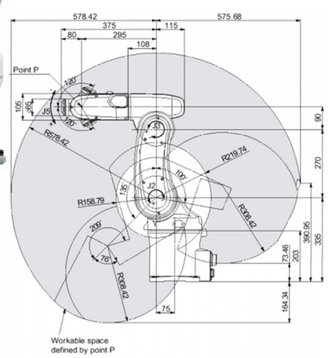
common configurations

coordinates

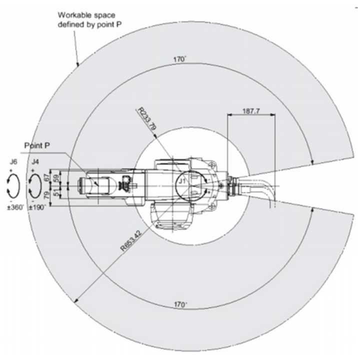
relative vs. absolute joints



Review: Workspace
Adept S850 Workspace



Side Dimensions and Work Envelope



Top Dimensions and Work Envelope

Review: Topics Covered

frame notation

2-d rotation matrix derivation

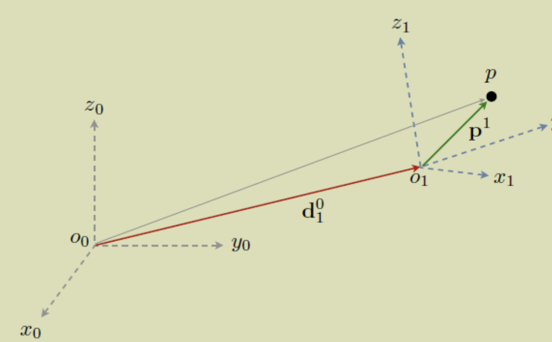
3-d rotation matrices

composition of rotations

Euler angles

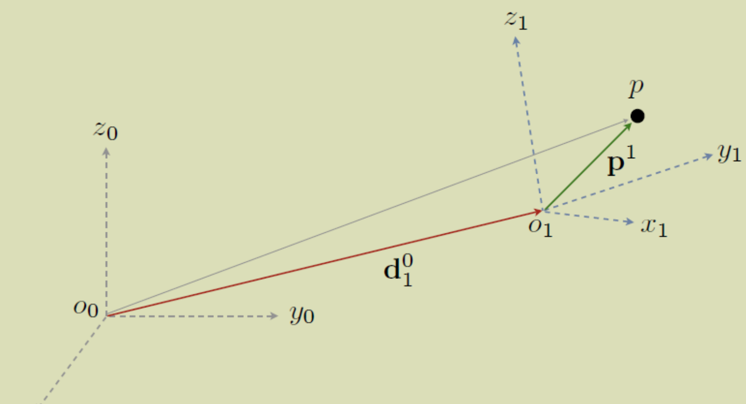
Roll, Pitch, Yaw angles

Axis/Angle representation



Review: Rigid Motions

a **rigid motion** couples pure translation with pure rotation



rigid motions can be expressed as

$$\mathbf{p}^0 = \mathbf{R}_1^0\,\mathbf{p}^1 + \mathbf{d}_1^0$$

Review: Homogeneous Transformations

a **homogeneous transform** is a matrix representation of rigid motion, defined as

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{d} \\ \mathbf{0} & 1 \end{bmatrix}$$

where $\mathbf{R}$ is the 3x3 rotation matrix, and $\mathbf{d}$ is the 1x3 translation vector

$$\mathbf{H} = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the **inverse** of a homogeneous transform can be expressed as

$$\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top d \\ 0 & 1 \end{bmatrix}$$

Review: Homogeneous Transformations

the **homogeneous representation** of a vector is formed by concatenating the original vector with a unit scalar

$$\mathbf{P} = \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix}$$

where $\mathbf{p}$ is the 1x3 vector

$$\mathbf{P} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Review: Homogeneous Transformations

rigid body transformations are accomplished by pre-multiplying by the homogenous transform

$$\mathbf{P}^0 = \mathbf{H}_1^0\,\mathbf{P}^1$$

composition of multiple transforms is the same as for rotation matrices:

post-multiply when successive rotations are relative to intermediate frames

$$\mathbf{H}_2^0 = \mathbf{H}_1^0\,\mathbf{H}_2^1$$

pre-multiply when successive rotations are relative to the first fixed frame

$$\mathbf{H}_2^0 = \mathbf{H}\,\mathbf{H}_1^0$$

Review: Homogeneous Transformations

Composition (intermediate frame)

$$\mathbf{H}_2^0 = \mathbf{H}_1^0\,\mathbf{H}_2^1 = \begin{bmatrix} \mathbf{R}_1^0 & \mathbf{d}_1^0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_2^1 & \mathbf{d}_2^1 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_2^0 & \mathbf{R}_1^0\mathbf{d}_2^1 + \mathbf{d}_1^0 \\ \mathbf{0} & 1 \end{bmatrix}$$

Inverse Transform

$$\mathbf{H}_0^1 = \begin{bmatrix} \mathbf{R}_0^1 & \mathbf{d}_0^1 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} (\mathbf{R}_1^0)^\top & -(\mathbf{R}_1^0)^\top\mathbf{d}_1^0 \\ \mathbf{0} & 1 \end{bmatrix}$$

Review: Homogeneous Transformations

- H that represents the following in order:
  - Rotation by angle α about current x-axis
  - Translation of b units along current x-axis
  - Translation of d units along current z-axis
  - Rotation by angle θ about current z-axis

$$H = \quad Rot_{x,\alpha}\; Trans_{x,\beta}\; Trans_{z,d}\; Rot_{z,\theta}$$

$$H = \begin{bmatrix} c_\theta & -s_\theta & 0 & \beta \\ c_\alpha s_\theta & c_\alpha & -s_\alpha & -ds_\theta \\ s_\alpha s_\theta & s_\alpha & c_\alpha & dc_\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Characterized by 6 numbers

Forward Kinematics
Denavit-Hartenberg Parameters

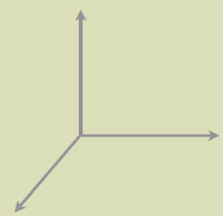
DH Parameters: Denavit-Hartenberg Convention

The **Denavit-Hartenberg convention** defines four parameters and some rules to help characterize arbitrary kinematic chains

start by attaching a frame to each link:

the joint variable for link i+1 acts along/around z_i

the axis x_i is perpendicular to, and intersects z_{i-1}



the following conventions make this process easier:

if z_{i-1} is parallel to z_i	orient x_i along normal with z_{i-1}
if z_{i-1} intersects z_i	orient x_i normal to the plane formed by z_{i-1} and z_i
if z_{i-1} is not coplanar with z_i	orient x_i along normal with z_{i-1}

Denavit & Hartenberg, "A kinematic notation for lower-pair mechanisms based on matrices," ASME Journal of Applied Mechanics, June 1955

DH Parameters: Denavit-Hartenberg Convention

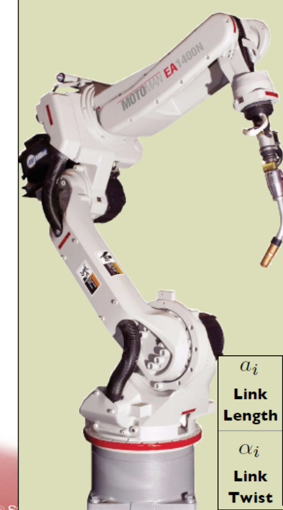
The **Denavit-Hartenberg convention** defines four parameters and some rules to help characterize arbitrary kinematic chains

a_i **Link Length**	the distance perpendicular to z_i and z_{i-1}, measured along x_i
α_i **Link Twist**	the angle between z_{i-1} and z_i, measured in the plane normal to x_i (right-hand rule around x_i)
d_i **Link Offset**	the distance along z_{i-1} from o_{i-1} to the intersection with x_i
θ_i **Joint Angle**	the angle between x_{i-1} and x_i, measured in the plane normal to z_{i-1} (right-hand rule around z_{i-1})

Denavit & Hartenberg, "A kinematic notation for lower-pair mechanisms based on matrices," ASME Journal of Applied Mechanics, June 1955

DH Parameters: Denavit-Hartenberg Transform

The **Denavit-Hartenberg transform** results from successive rotations and translations via the four DH parameters

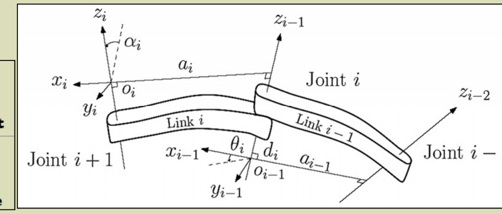


The transform from i-1 to i:

$$A_i = \text{Rot}_{z,\theta_i}\ \text{Trans}_{z,d_i}\ \text{Trans}_{x,a_i}\ \text{Rot}_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a_i Link Length	d_i Link Offset
α_i Link Twist	θ_i Joint Angle



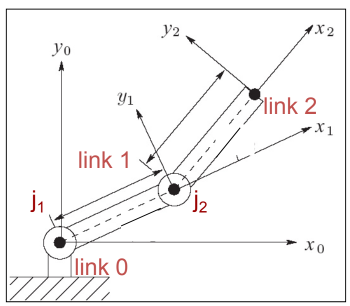
DH Parameters: Example 1- Planar Elbow Manipulator

Link and Joint Labeling Scheme:

\# of joints = n Number joints from 1 to *n*

\# of links = n + 1 Number links from 0 to *n*

When joint i is actuated, link i moves → Link 0 is fixed



start by attaching a frame to each link:

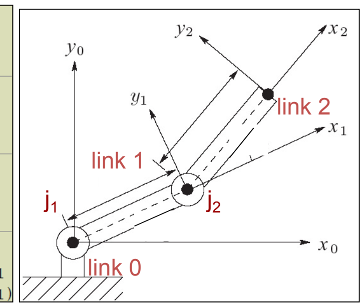
the joint variable for link i+1 acts along/around z_i

the axis x_i is perpendicular to, and intersects z_{i-1}

if z_{i-1} is parallel to z_i	orient x_i along normal with z_{i-1}
if z_{i-1} intersects z_i	orient x_i normal to the plane formed by z_{i-1} and z_i
if z_{i-1} is not coplanar with z_i	orient x_i along normal with z_{i-1}

DH Parameters: Example 1- Planar Elbow Manipulator

a_i **Link Length**	the distance perpendicular to z_i and z_{i-1}, measured along x_i
α_i **Link Twist**	the angle between z_{i-1} and z_i, measured in the plane normal to x_i (right-hand rule around x_i)
d_i **Link Offset**	the distance along z_{i-1} from o_{i-1} to the intersection with x_i
θ_i **Joint Angle**	the angle between x_{i-1} and x_i, measured in the plane normal to z_{i-1} (right-hand rule around z_{i-1})



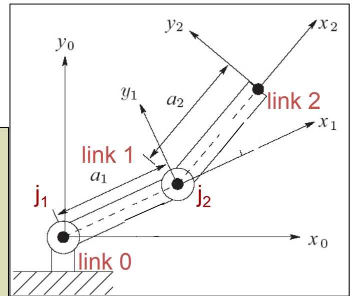
Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2

DH Parameters: Example 1- Planar Elbow Manipulator

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2

$$A_i = \mathrm{Rot}_{z,\theta_i}\,\mathrm{Trans}_{z,d_i}\,\mathrm{Trans}_{x,a_i}\,\mathrm{Rot}_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



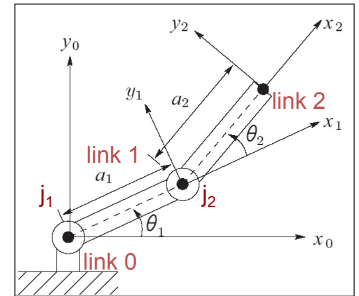
$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DH Parameters: Example 1- Planar Elbow Manipulator

$$H = T_n^0 = A_1(q_1)\dots A_n(q_n)$$

$$T_1^0 = A_1$$

$$T_2^0 = A_1A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Components of the origin o_2 in the base frame
→ end-effector coordinates

$x = a_1c_1 + a_2c_{12}$

$y = a_1s_1 + a_2s_{12}$

$z = 0$

Orientation of frame $o_2x_2y_2z_2$ relative to the base frame

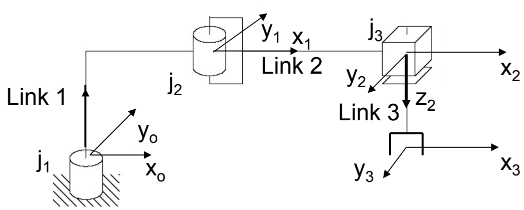
DH Parameters: Example 2a- SCARA Manipulator

start by attaching a frame to each link:

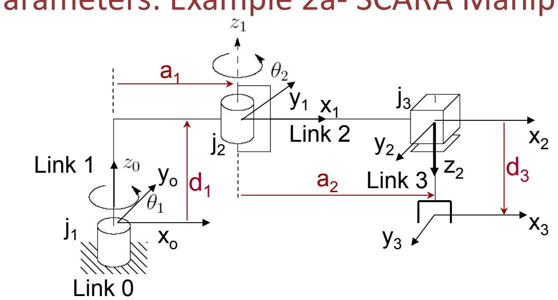
the joint variable for link i+1 acts along/around z_i

the axis x_i is perpendicular to, and intersects z_{i-1}

if z_{i-1} is parallel to z_i	orient x_i along normal with z_{i-1}
if z_{i-1} intersects z_i	orient x_i normal to the plane formed by z_{i-1} and z_i
if z_{i-1} is not coplanar with z_i	orient x_i along normal with z_{i-1}



DH Parameters: Example 2a- SCARA Manipulator



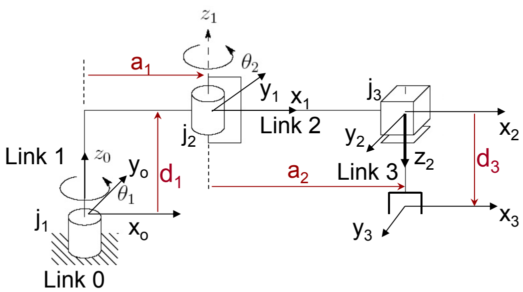
a_i **Link Length**	the distance perpendicular to z_i and z_{i-1}, measured along x_i
α_i **Link Twist**	the angle between z_{i-1} and z_i, measured in the plane normal to x_i (right-hand rule around x_i)
d_i **Link Offset**	the distance along z_{i-1} from O_{i-1} to the intersection with x_i
θ_i **Joint Angle**	the angle between x_{i-1} and x_i, measured in the plane normal to z_{i-1} (right-hand rule around z_{i-1})

Link	a_i	α_i	d_i	θ_i
1	a_1	0	d_1	θ_1*
2	a_2	180	0	θ_2*
3	0	0	d_3*	0

DH Parameters: Example 2a- SCARA Manipulator

Link	a_i	α_i	d_i	θ_i
1	a_1	0	d_1	θ_1*
2	a_2	180	0	θ_2*
3	0	0	d_3*	0

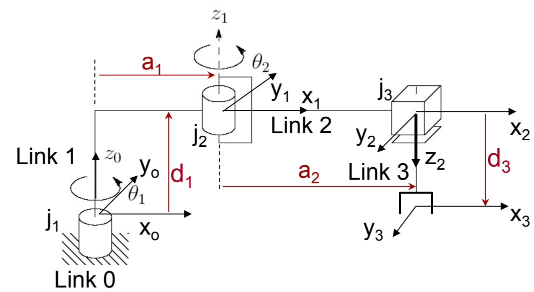
$$A_i = \mathrm{Rot}_{z,\theta_i}\,\mathrm{Trans}_{z,d_i}\,\mathrm{Trans}_{x,a_i}\,\mathrm{Rot}_{x,\alpha_i} = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



[Matlab Symbolic Math Toolbox]

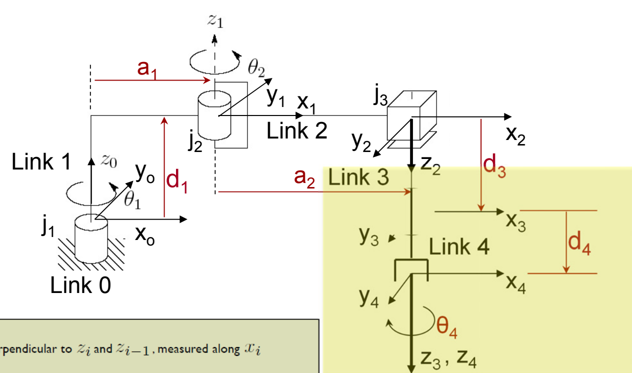
$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DH Parameters: Example 2a- SCARA Manipulator



$$T_3^0 = A_1A_2A_3 = \begin{bmatrix} c_{12} & s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & -c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & d_1 - d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

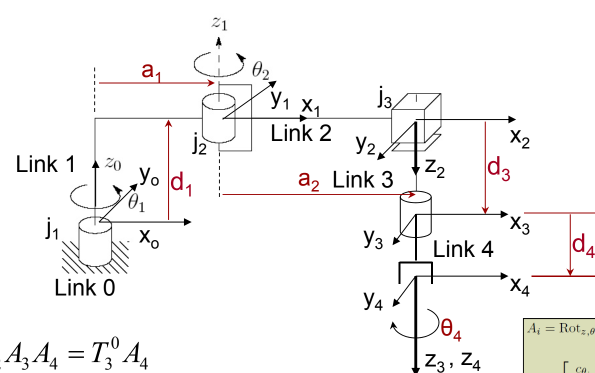
DH Parameters: Example 2b- SCARA Manipulator



a_i **Link Length**	the distance perpendicular to z_i and z_{i-1}, measured along x_i
α_i **Link Twist**	the angle between z_{i-1} and z_i, measured in the plane normal to x_i (right-hand rule around x_i)
d_i **Link Offset**	the distance along z_{i-1} from O_{i-1} to the intersection with x_i
θ_i **Joint Angle**	the angle between x_{i-1} and x_i, measured in the plane normal to z_{i-1} (right-hand rule around z_{i-1})

Link	a_i	α_i	d_i	θ_i
1	a_1	0	d_1	θ_1*
2	a_2	180	0	θ_2*
3	0	0	d_3*	0
4	0	0	d_4	θ_4

DH Parameters: Example 2b- SCARA Manipulator

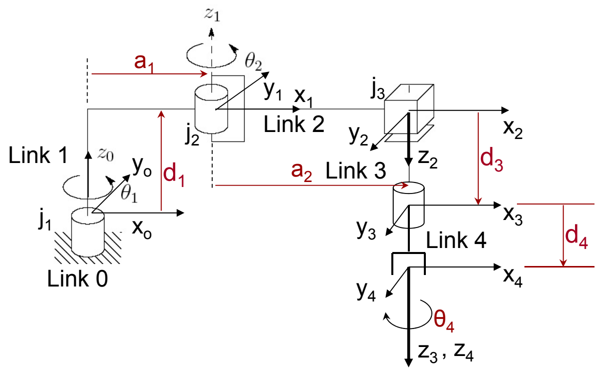


$$A_i = \mathrm{Rot}_{z,\theta_i}\,\mathrm{Trans}_{z,d_i}\,\mathrm{Trans}_{x,a_i}\,\mathrm{Rot}_{x,\alpha_i} = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4^0 = A_1A_2A_3A_4 = T_3^0A_4$$

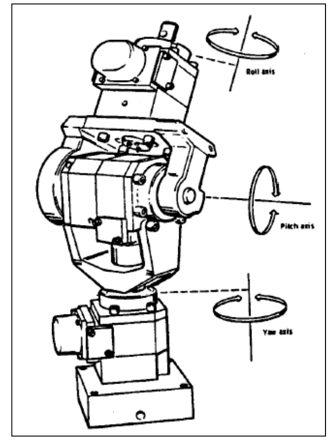
$$= \begin{bmatrix} c_{12} & s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & -c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & d_1 - d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

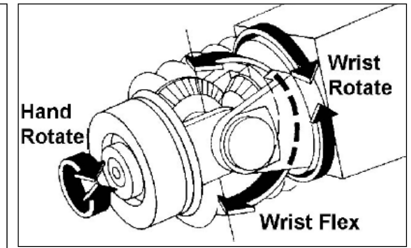
DH Parameters: Example 2b- SCARA Manipulator



$$T_4^0 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & d_1 - d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DH Parameters: Example 3- Spherical Wrist

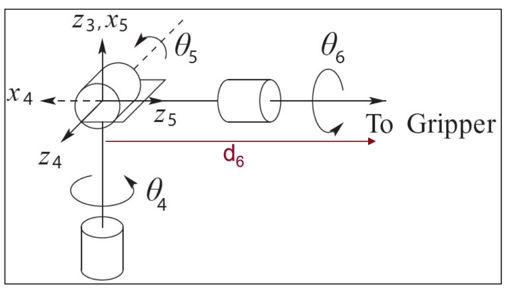




Three revolute joints intersecting at a common point

DH Parameters: Example 3- Spherical Wrist

Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4*
5	0	90	0	θ_5*
6	0	0	d_6	θ_6*



$$T_6^3 = A_4A_5A_6$$

$$= \begin{bmatrix} R_6^3 & o_6^3 \\ 0 & 1 \end{bmatrix}$$

$$T_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5c_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5c_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DH Parameters: Example 3- Spherical Wrist

$$T_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5c_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5c_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Euler Angles to Rotation Matrices

$$\mathbf{R} = \mathbf{R}_{z,\phi}\,\mathbf{R}_{y,\theta}\,\mathbf{R}_{z,\psi}$$

$$= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

θ_4, θ_5, and θ_6 are the Euler angles Φ, θ, and ψ with respect to the coordinate frame $o_3x_3y_3z_3$ → will help with Inverse Kinematics

DH Parameters: Euler Angles from Spherical Wrist Eqs

$$\begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

To find a solution for this problem we break it down into two cases. First, suppose that not both of r_{13}, r_{23} are zero. Then from Equation (2.26) we deduce that $s_\theta \neq 0$, and hence that not both of r_{31}, r_{32} are zero. If not both r_{13} and r_{23} are zero, then $r_{33} \neq \pm 1$, and we have $c_\theta = r_{33}$, $s_\theta = \pm\sqrt{1 - r_{33}^2}$ so

$$\theta = \text{Atan2}\left(r_{33}, \sqrt{1 - r_{33}^2}\right) \quad (2.28)$$

or

$$\theta = \text{Atan2}\left(r_{33}, -\sqrt{1 - r_{33}^2}\right) \quad (2.29)$$

where the function Atan2 is the **two-argument arctangent function** defined in Appendix A.

If we choose the value for θ given by Equation (2.28), then $s_\theta > 0$, and

$$\phi = \text{Atan2}(r_{13}, r_{23}) \quad (2.30)$$
$$\psi = \text{Atan2}(-r_{31}, r_{32}) \quad (2.31)$$

If we choose the value for θ given by Equation (2.29), then $s_\theta < 0$, and

$$\phi = \text{Atan2}(-r_{13}, -r_{23}) \quad (2.32)$$
$$\psi = \text{Atan2}(r_{31}, -r_{32}) \quad (2.33)$$

Thus, there are two solutions depending on the sign chosen for θ.

[Spong et. al., pages 55-56]

DH Parameters: Euler Angles from Spherical Wrist Eqs

If $r_{13} = r_{23} = 0$, then the fact that R is orthogonal implies that $r_{33} = \pm 1$, and that $r_{31} = r_{32} = 0$. Thus, R has the form

$$R = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \quad (2.34)$$

If $r_{33} = 1$, then $c_\theta = 1$ and $s_\theta = 0$, so that $\theta = 0$. In this case, Equation (2.26) becomes

$$\begin{bmatrix} c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi & 0 \\ s_\phi c_\psi + c_\phi s_\psi & -s_\phi s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the sum $\phi + \psi$ can be determined as

$$\phi + \psi = \text{Atan2}(r_{11}, r_{21}) = \text{Atan2}(r_{11}, -r_{12}) \quad (2.35)$$

Since only the sum $\phi + \psi$ can be determined in this case, there are infinitely many solutions. In this case, we may take $\phi = 0$ by convention. If $r_{33} = -1$, then $c_\theta = -1$ and $s_\theta = 0$, so that $\theta = \pi$. In this case Equation (2.26) becomes

$$\begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (2.36)$$

The solution is thus

$$\phi - \psi = \text{Atan2}(-r_{11}, -r_{12}) \quad (2.37)$$

As before there are infinitely many solutions.

[Spong et. al., pages 55-56]

Inverse Kinematics

Inverse Kinematics

- Given end-effector position and orientation, compute corresponding joint variables

Algebraic Decomposition

Geometric Analysis

Inverse Kinematics: Algebraic Decomposition

given the forward transform matrix for a manipulator

$$\mathbf{T}_n^0 = \begin{bmatrix} [\mathbf{R}_n^0(\mathbf{q})]_{3\times 3} & [\mathbf{d}_n^0(\mathbf{q})]_{3\times 1} \\ [\mathbf{0}]_{1\times 3} & 1 \end{bmatrix}$$

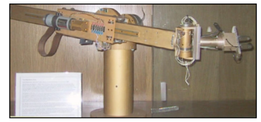
solve the system of 3 equations from the displacement vector

$$\begin{aligned} d_x &= [\mathbf{d}_n^0(\mathbf{q})]_1 \\ d_y &= [\mathbf{d}_n^0(\mathbf{q})]_2 \\ d_z &= [\mathbf{d}_n^0(\mathbf{q})]_3 \end{aligned}$$

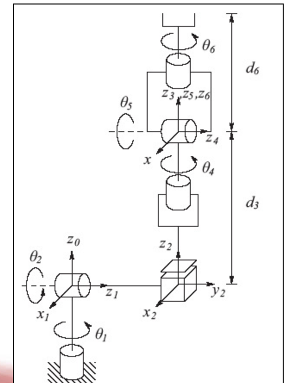
to find the joint variables in terms of the end-effector position

$$\mathbf{q} = \begin{bmatrix} q_1(d_x, d_y, d_z) \\ q_2(d_x, d_y, d_z) \\ \vdots \\ q_n(d_x, d_y, d_z) \end{bmatrix}$$

Inverse Kinematics: Algebraic Decomposition



Link	d_i	a_i	α_i	θ_i
1	0	0	−90	$\theta_1^\star$
2	d_2	0	+90	$\theta_2^\star$
3	$d_3^\star$	0	0	0
4	0	0	−90	$\theta_4^\star$
5	0	0	+90	$\theta_5^\star$
6	d_6	0	0	$\theta_6^\star$



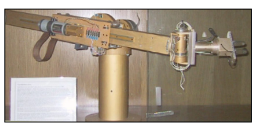
T_6^0 is then given as

$$T_6^0 = A_1 \cdots A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

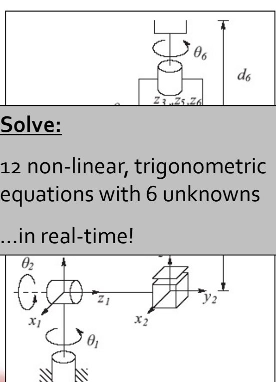
in which

$$\begin{aligned} r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) \\ r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\ r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\ r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\ r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) \\ r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\ r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\ r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\ r_{33} &= -s_2c_4s_5 + c_2c_5 \\ d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\ d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\ d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) \end{aligned}$$

Inverse Kinematics: Algebraic Decomposition



Link	d_i	a_i	α_i	θ_i
1	0	0	−90	$\theta_1^\star$
2	d_2	0	+90	$\theta_2^\star$
3	$d_3^\star$	0	0	0
4	0	0	−90	$\theta_4^\star$
5	0	0	+90	$\theta_5^\star$
6	d_6	0	0	$\theta_6^\star$



Solve:

12 non-linear, trigonometric equations with 6 unknowns

...in real-time!

Example 3.7

Recall the Stanford manipulator of Example 3.5. Suppose that the desired position and orientation of the final frame are given by

$$H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3.29)$$

To find the corresponding joint variables θ_1, θ_2, d_3, θ_4, θ_5, and θ_6 we must solve the following simultaneous set of nonlinear trigonometric equations:

$$\begin{aligned} c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= 0 \\ s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0 \\ -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1 \\ c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1 \\ s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= 0 \\ s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0 \\ c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0 \\ s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= 1 \\ -s_2c_4s_5 + c_2c_5 &= 0 \\ c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= -0.154 \\ s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763 \\ c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0 \end{aligned}$$

Inverse Kinematics: Geometric Analysis

For most simple manipulators, it is often easier to use geometry to solve for closed-form solutions to the inverse kinematics

solve for each joint variable q_i by projecting the manipulator onto the x_{i-1}, y_{i-1} plane

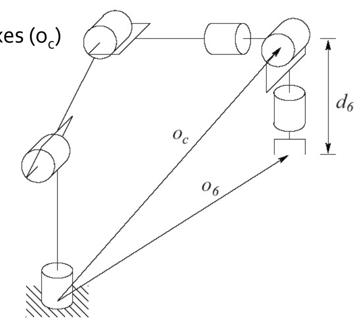
closed-form inverse kinematic solutions are not always possible, and if it is solvable, there are often multiple solutions

Inverse Kinematics: Kinematic Decoupling

Inverse kinematics =
inverse position kinematics +
inverse orientation kinematics

Two sub problems:

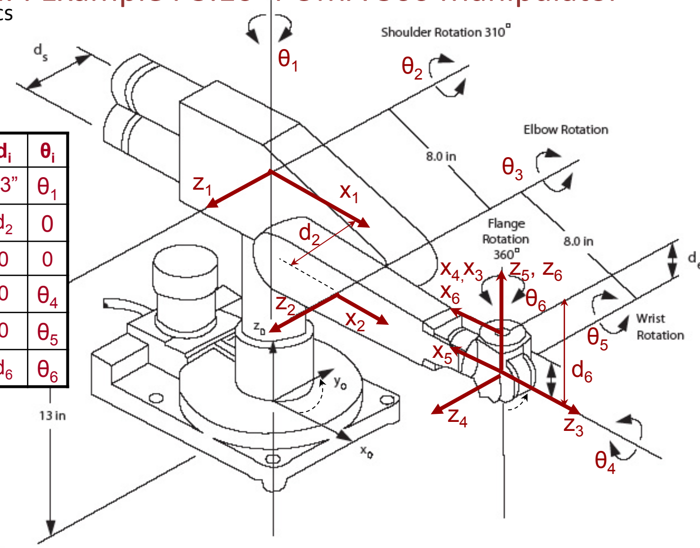
- Find position of the intersection of the wrist axes (o_c)
- Find orientation of the wrist



Inv. Kin. : Example P3.10- PUMA 360 Manipulator

Forward Kinematics

Link	a_i	α_i	d_i	θ_i
1	0	90	13"	θ_1
2	8"	0	d_2	0
3	8"	90	0	0
4	0	-90	0	θ_4
5	0	90	0	θ_5
6	0	0	d_6	θ_6

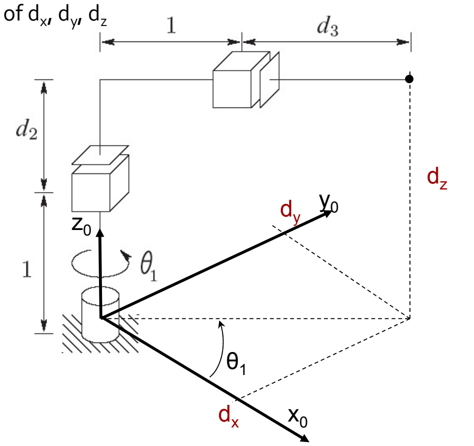


Inv. Kin. : Example P3.13- Cylindrical Manipulator

Inverse Kinematics

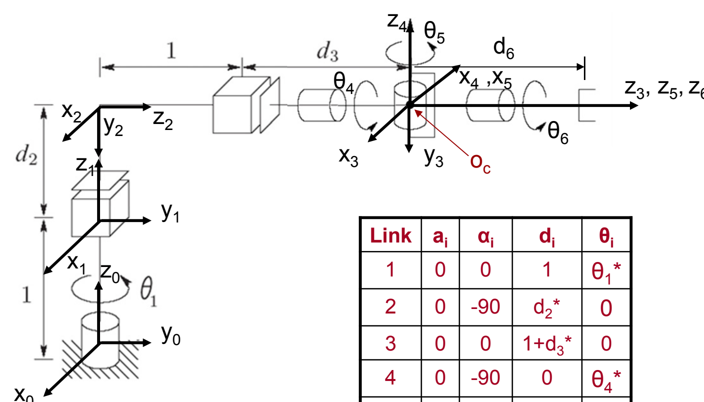
Given: $d = [d_x, d_y, d_z]^T$

Find: θ_1, d_2, and d_3 as functions of d_x, d_y, d_z



solve for each joint variable q_i by projecting the manipulator onto the x_{i-1}, y_{i-1} plane

Inv. Kin. : Example P3.15- Cylindrical Manipulator + Spherical Wrist



Link	a_i	α_i	d_i	θ_i
1	0	0	1	θ_1*
2	0	-90	d_2*	0
3	0	0	1+d_3*	0
4	0	-90	0	θ_4*
5	0	90	0	θ_5*
6	0	0	d_6	θ_6*

Summary Algorithms for Denavit-Hartenberg Parameters

Summary Algorithms for DH Parameters: Forward Kinematics

[Spong et. al., pages 110-111]

Step 1: Locate and label the joint axes $z_0, \ldots, z_{n-1}$.

Step 2: Establish the base frame. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-handed frame.

For $i = 1, \ldots, n-1$ perform Steps 3 to 5.

Step 3: Locate the origin o_i where the common normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1} locate o_i at this intersection. If z_i and z_{i-1} are parallel, locate o_i in any convenient position along z_i.

Step 4: Establish x_i along the common normal between z_{i-1} and z_i through o_i, or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

Step 5: Establish y_i to complete a right-handed frame.

Step 6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n^{th} joint is revolute, set $z_n = a$ parallel to z_{n-1}. Establish the origin o_n conveniently along z_n, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = s$ in the direction of the gripper closure and set $x_n = n$ as $s \times a$. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-handed frame.

Summary Algorithms for DH Parameters: Forward Kinematics

[Spong et. al., pages 110-111]

Step 7: Create a table of DH parameters a_i, d_i, α_i, θ_i.

$a_i =$ distance along x_i from the intersection of the x_i and z_{i-1} axes to o_i.

$d_i =$ distance along z_{i-1} from o_{i-1} to the intersection of the x_i and z_{i-1} axes. If joint i is prismatic, d_i is variable.

$\alpha_i =$ the angle from z_{i-1} to z_i measured about x_i.

$\theta_i =$ the angle from x_{i-1} to x_i measured about z_{i-1}. If joint i is revolute, θ_i is variable.

Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into Equation (3.10).

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

Summary Algorithms for DH Parameters: Inverse Kinematics (Manipulators w/Spherical Wrists)

[Spong et. al., pages 110-111]

Step 1: Find q_1, q_2, q_3 such that the wrist center o_c has coordinates given by

$$o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (3.79)$$

Step 2: Using the joint variables determined in Step 1, evaluate R_3^0.

Step 3: Find a set of Euler angles corresponding to the rotation matrix

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R \quad (3.80)$$

In this chapter we demonstrated a geometric approach for Step 1. In particular, to solve for joint variable q_i, we project the manipulator (including the wrist center) onto the $x_{i-1} - y_{i-1}$ plane and use trigonometry to find q_i.

Announcements

- Homework #2
- Manipulator Lab 1

Typo in book for Law of Cosines in Appendix A. Should be:

$$c^2 = a^2 + b^2 - 2\, a\, b \cos\theta$$