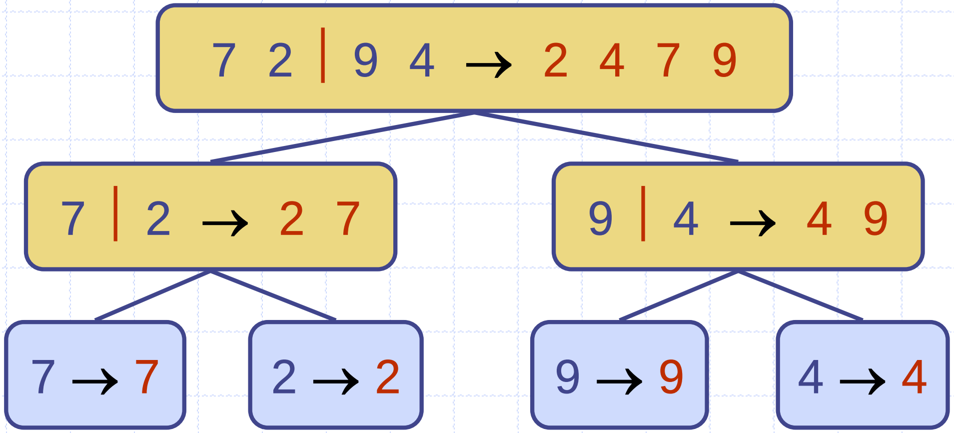
Merge Sort



Outline and Reading

- Divide-and-conquer paradigm, MergeSort (§4.1)
- Sets (§4.2);Generic Merging and set operations (§4.2.1)
  - Note: Sections 4.2.2 and 4.2.3 are Optional
- Quick-sort (§4.3)
- Analysis of quick-sort ((§4.3.1)
- A Lower Bound on Comparison-based Sorting (§4.4)
- QuickSort and Radix Sort (§4.5)
- In-place quick-sort (§4.8)
- Comparison of Sorting Algorithm (§4.6)
- Selection (§4.7)

Divide-and-Conquer

- Divide-and conquer is a general algorithm design paradigm:
  - Divide: divide the input data S in two disjoint subsets S_1 and S_2
  - Recur: solve the subproblems associated with S_1 and S_2
  - Conquer: combine the solutions for S_1 and S_2 into a solution for S
- The base case for the recursion are subproblems of size 0 or 1
- Merge-sort is a sorting algorithm based on the divide-and-conquer paradigm
- Like heap-sort
  - It uses a comparator
  - It has $O(n \log n)$ running time
- Unlike heap-sort
  - It does not use an auxiliary priority queue
  - It accesses data in a sequential manner (suitable to sort data on a disk)

Merge-Sort

- Merge-sort on an input sequence S with n elements consists of three steps:
  - Divide: partition S into two sequences S_1 and S_2 of about $n/2$ elements each
  - Recur: recursively sort S_1 and S_2
  - Conquer: merge S_1 and S_2 into a unique sorted sequence

Algorithm *mergeSort*(S, C)
    Input sequence S with n elements, comparator C
    Output sequence S sorted according to C
    if S.*size*() > 1
        $(S_1, S_2) \leftarrow$ *partition*(S, $n/2$)
        mergeSort(S_1, C)
        mergeSort(S_2, C)
        $S \leftarrow$ *merge*(S_1, S_2)

Merging Two Sorted Sequences

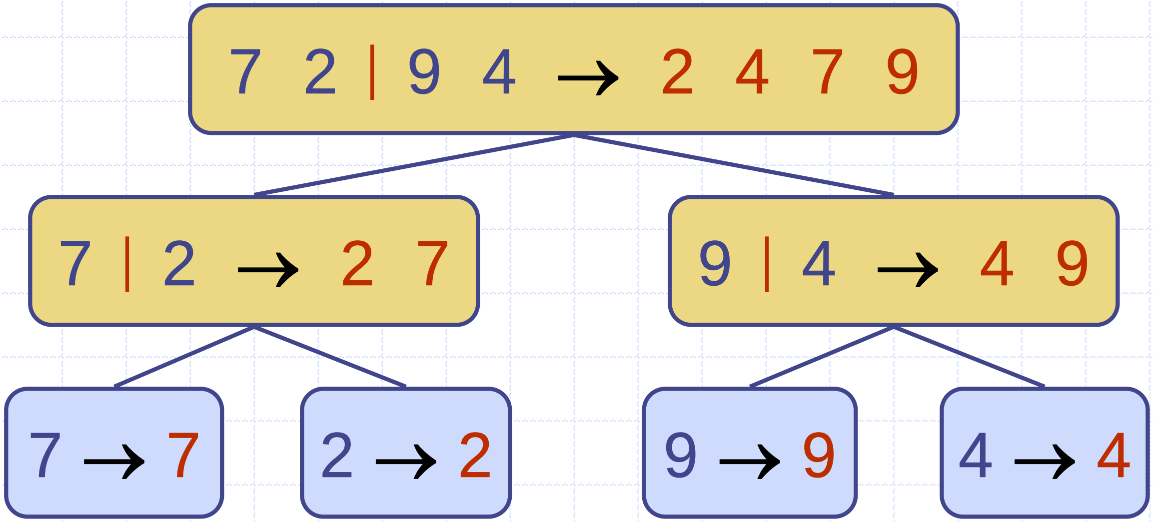
- The conquer step of merge-sort consists of merging two sorted sequences A and B into a sorted sequence S containing the union of the elements of A and B
- Merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes $O(n)$ time

Algorithm *merge*(A, B)
  Input sequences A and B with
    $n/2$ elements each
  Output sorted sequence of $A \cup B$

  $S \leftarrow$ empty sequence
  while $\neg A.isEmpty() \wedge \neg B.isEmpty()$
    if $A.first().element() < B.first().element()$
      $S.insertLast(A.remove(A.first()))$
    else
      $S.insertLast(B.remove(B.first()))$
  while $\neg A.isEmpty()$
    $S.insertLast(A.remove(A.first()))$
  while $\neg B.isEmpty()$
    $S.insertLast(B.remove(B.first()))$
  return S

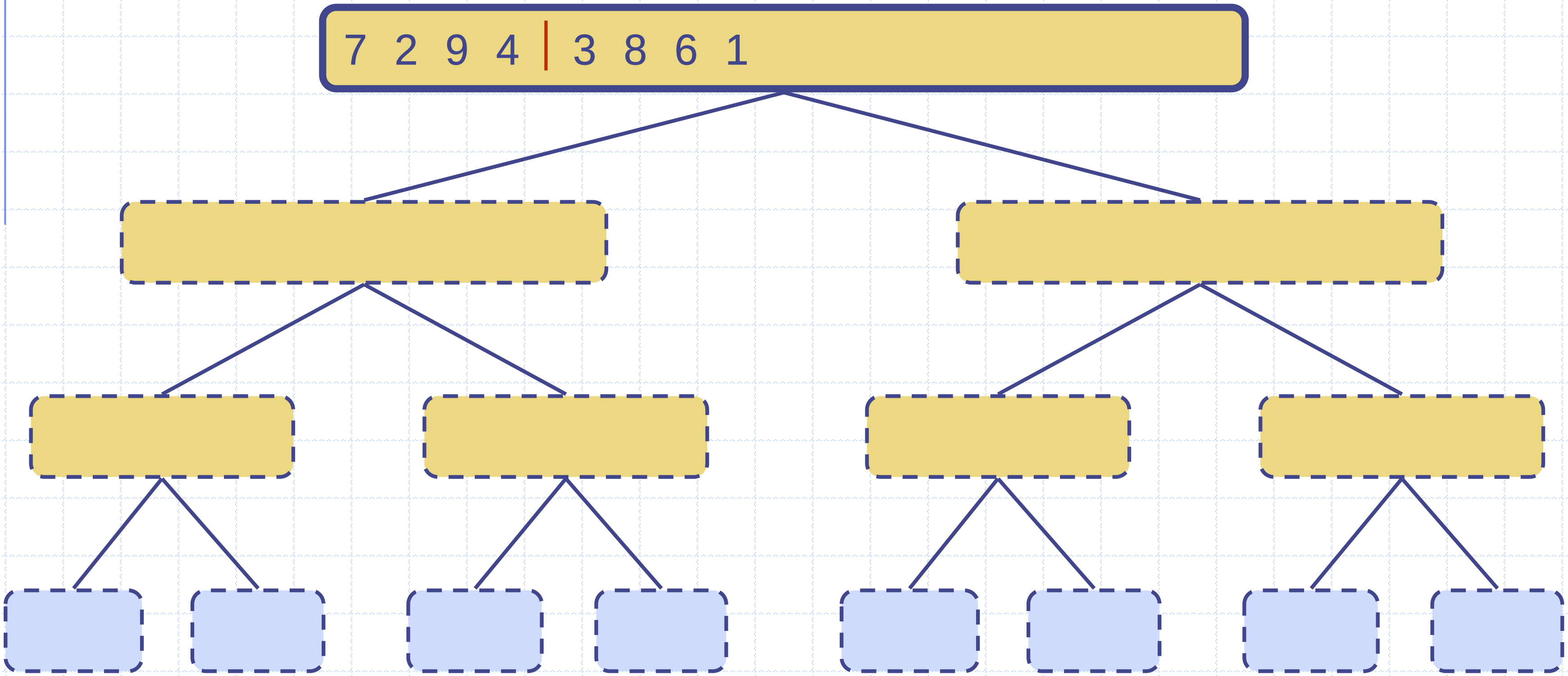
Merge-Sort Tree

- An execution of merge-sort is depicted by a binary tree
  - each node represents a recursive call of merge-sort and stores
    - unsorted sequence before the execution and its partition
    - sorted sequence at the end of the execution
  - the root is the initial call
  - the leaves are calls on subsequences of size 0 or 1



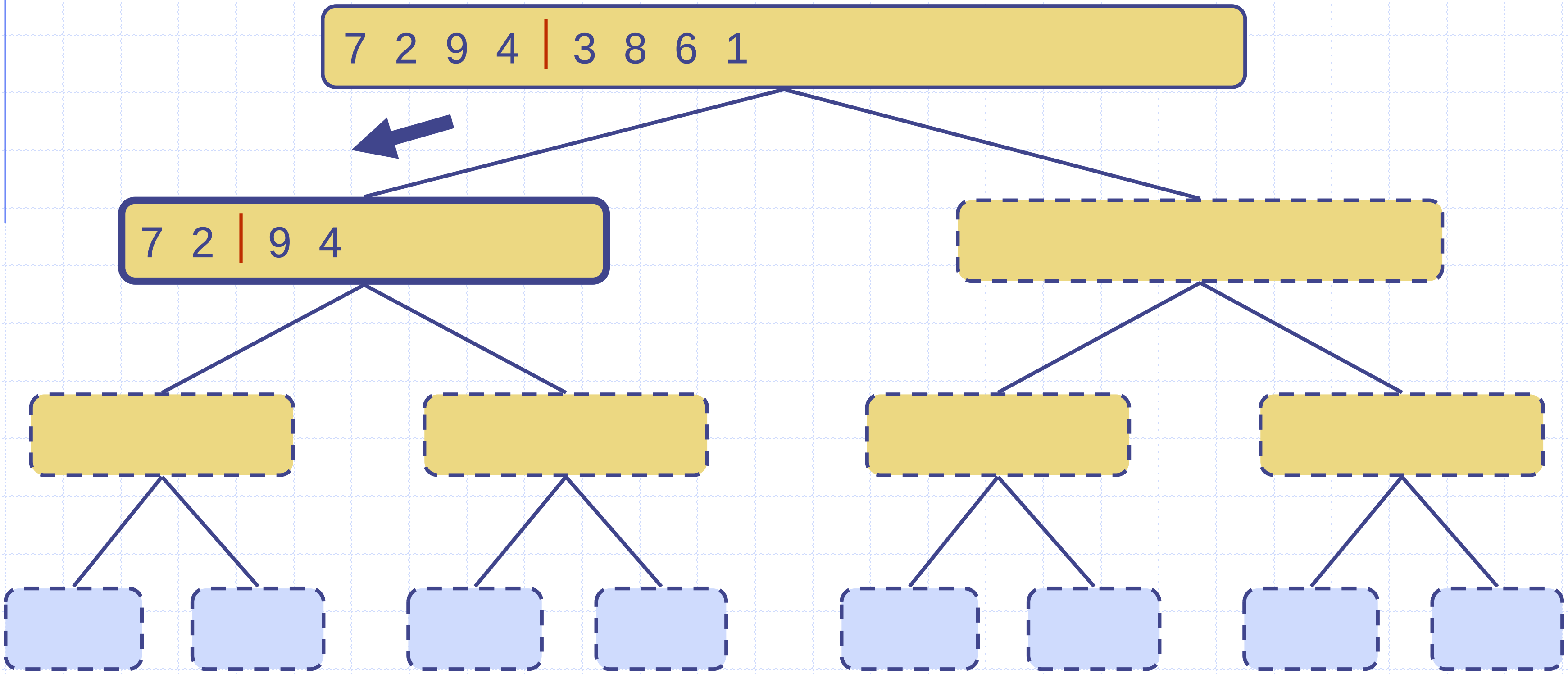
Execution Example

- Partition



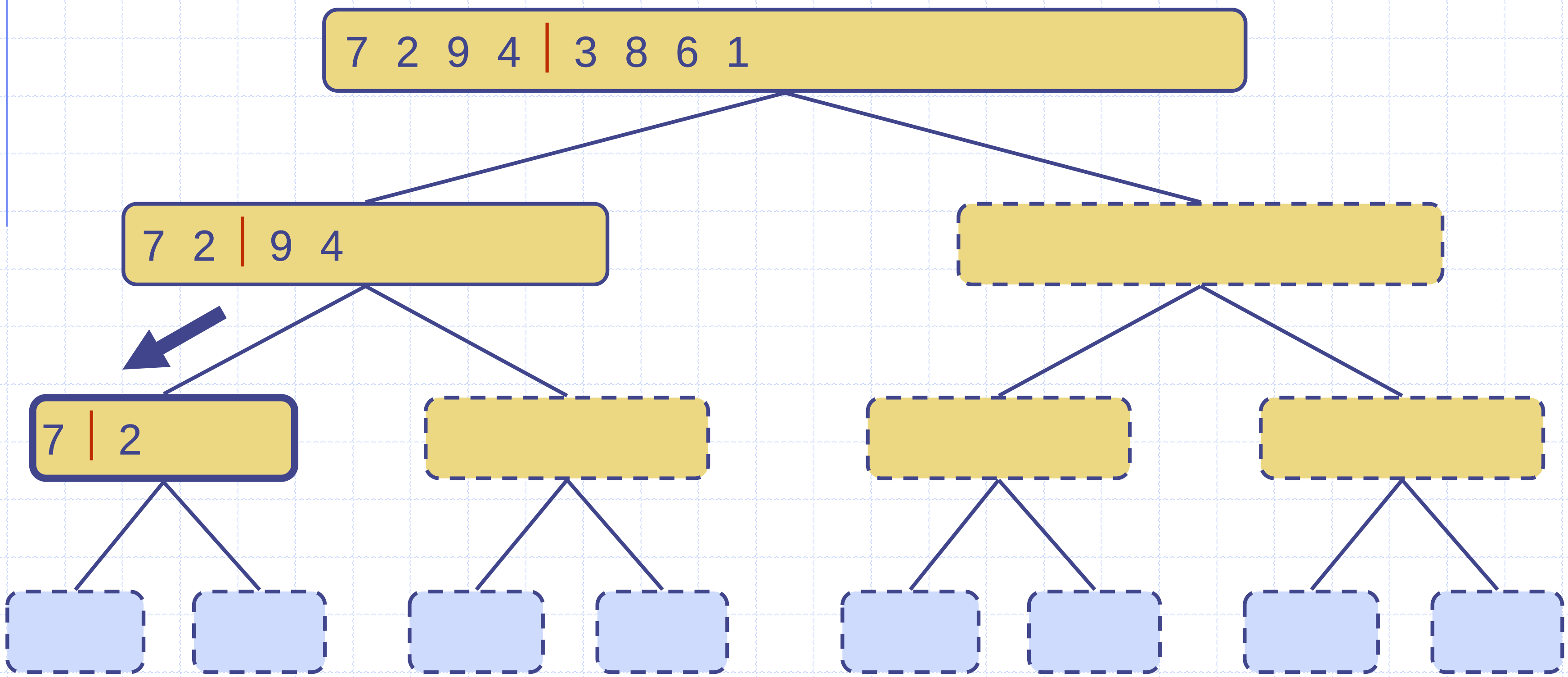
Execution Example (cont.)

- Recursive call, partition



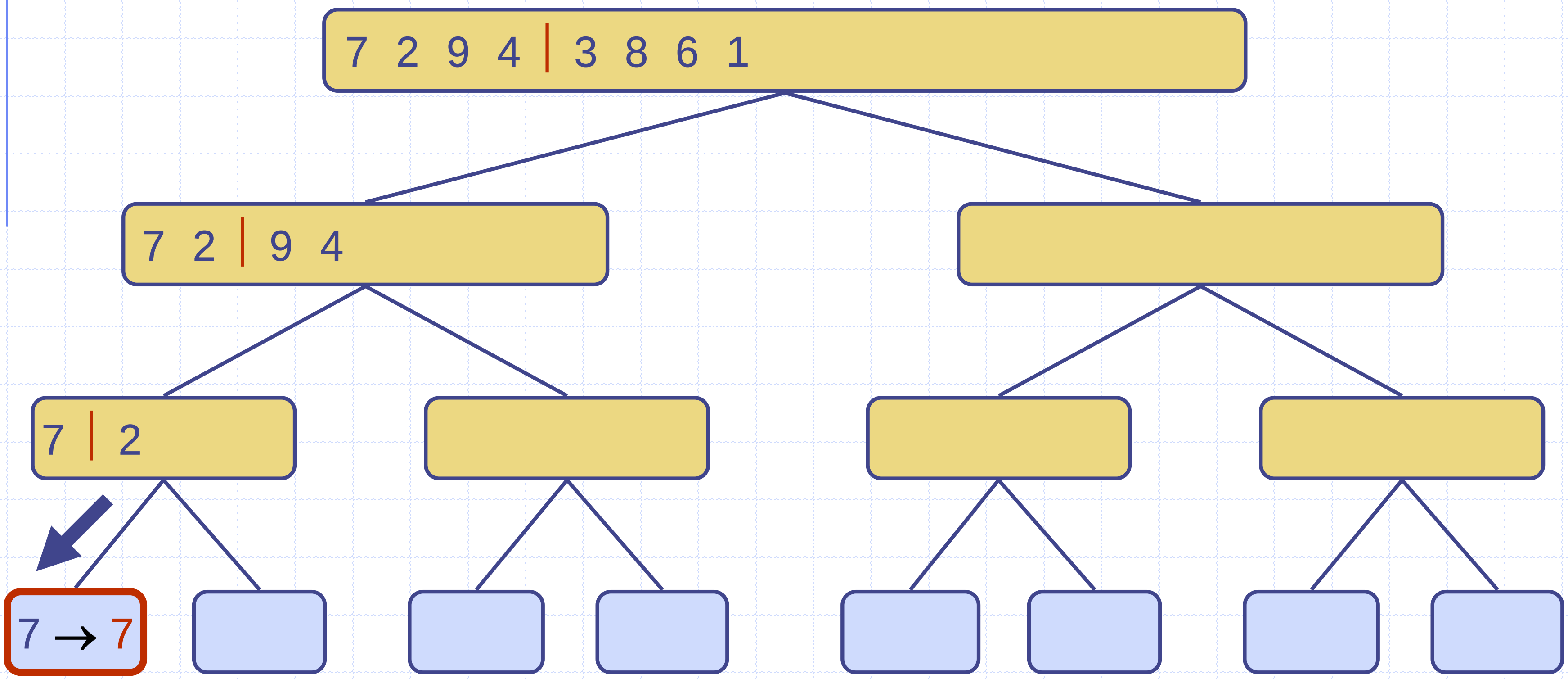
Execution Example (cont.)

- Recursive call, partition



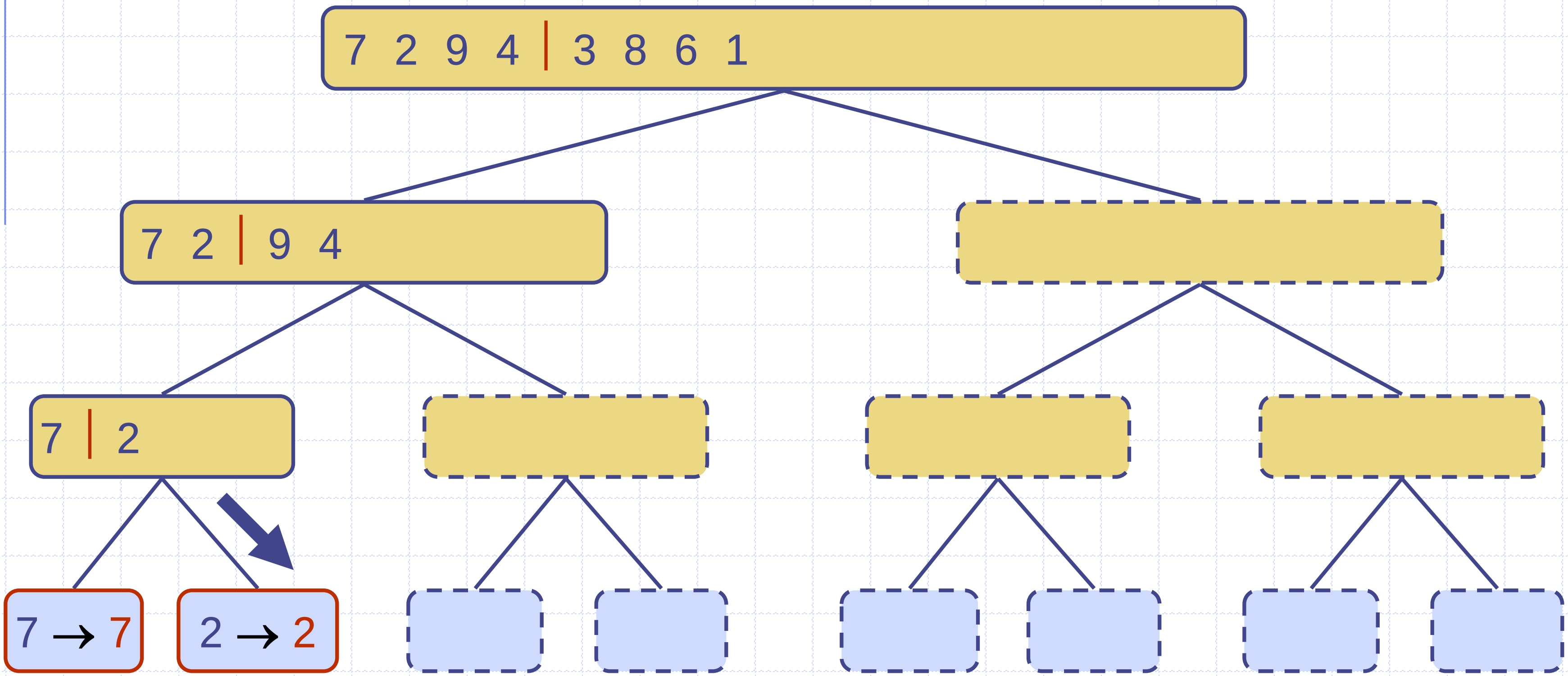
Execution Example (cont.)

- Recursive call, base case



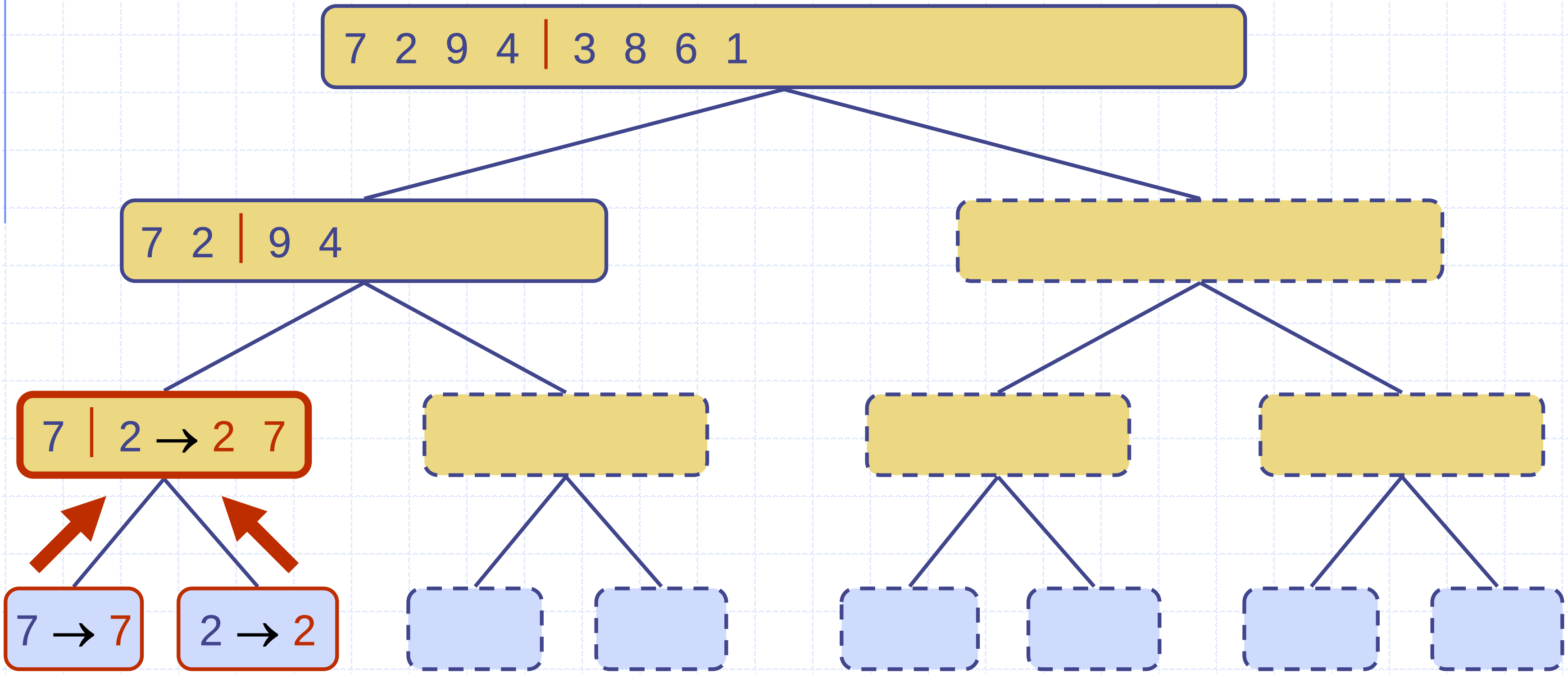
Execution Example (cont.)

- Recursive call, base case



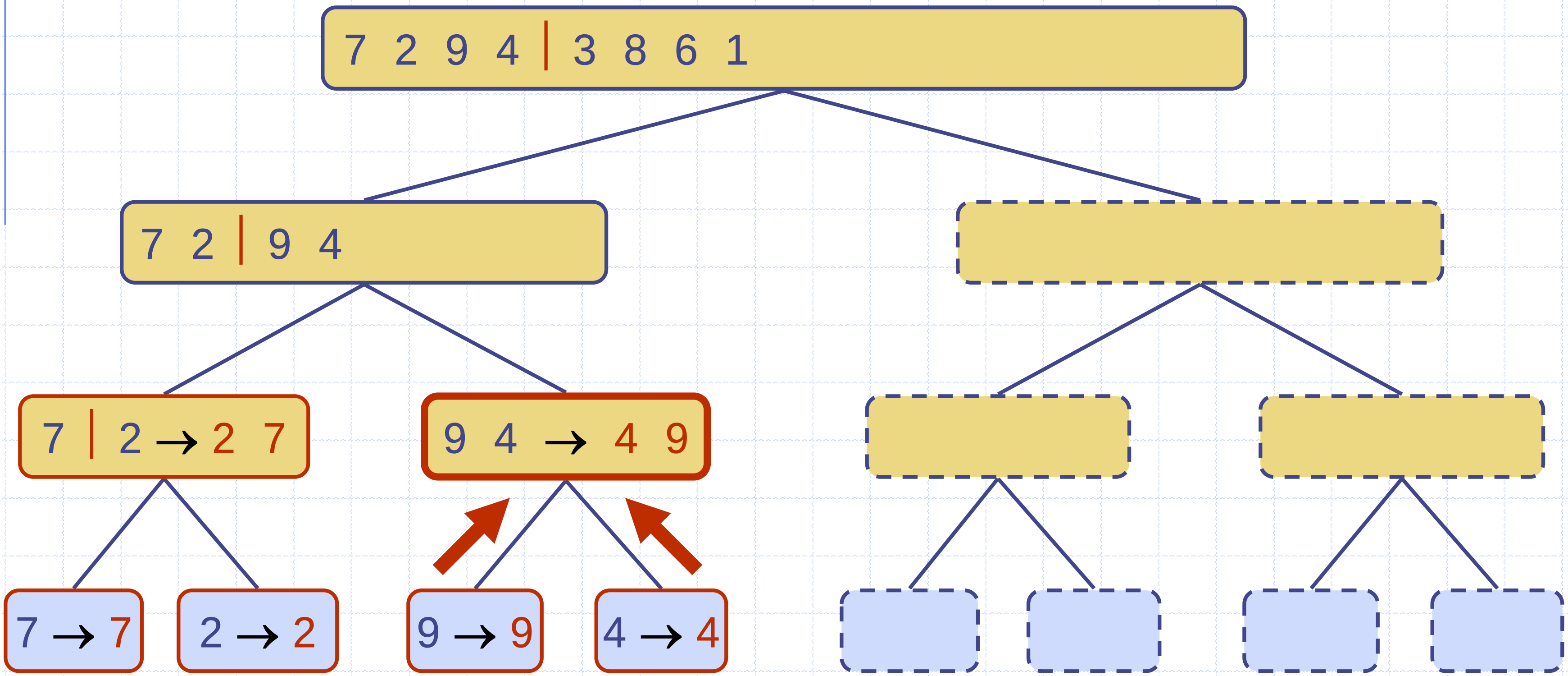
Execution Example (cont.)

- Merge



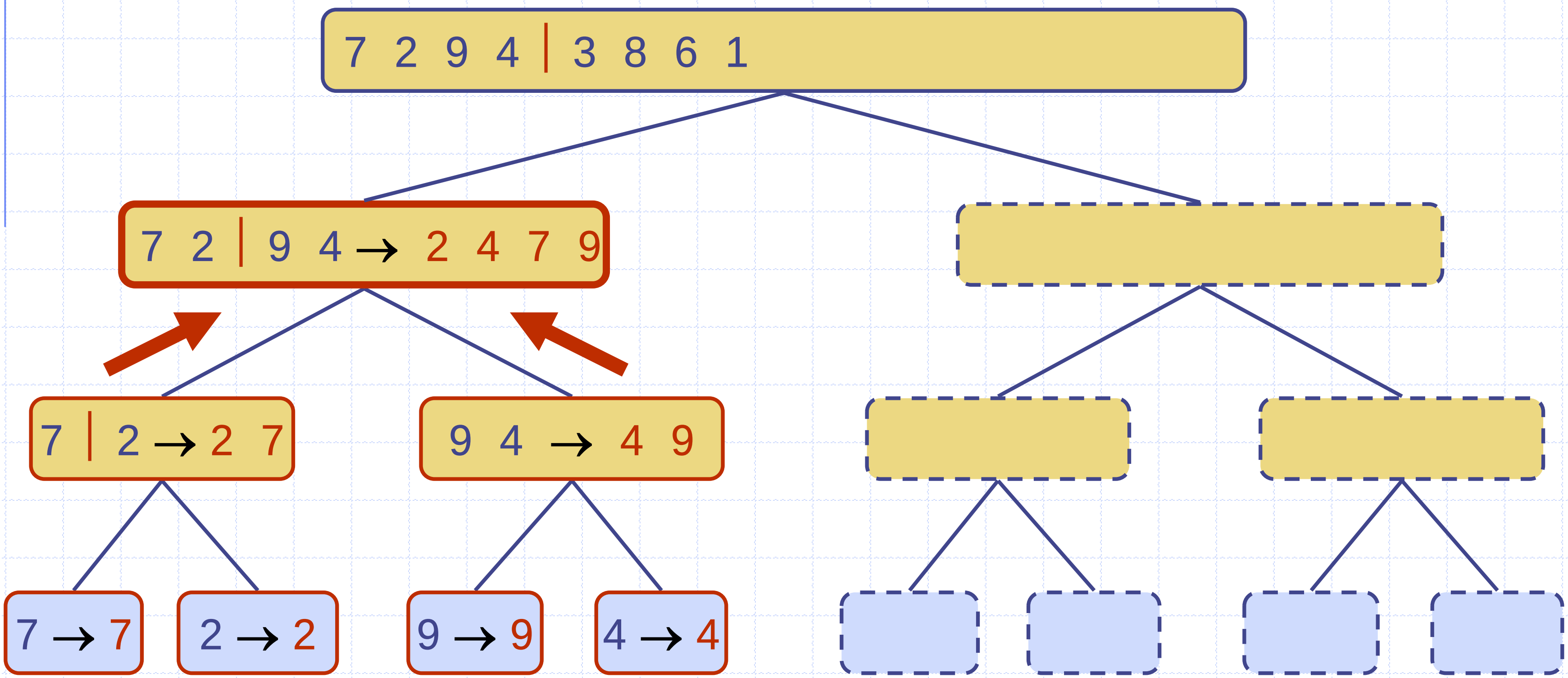
Execution Example (cont.)

- Recursive call, ..., base case, merge



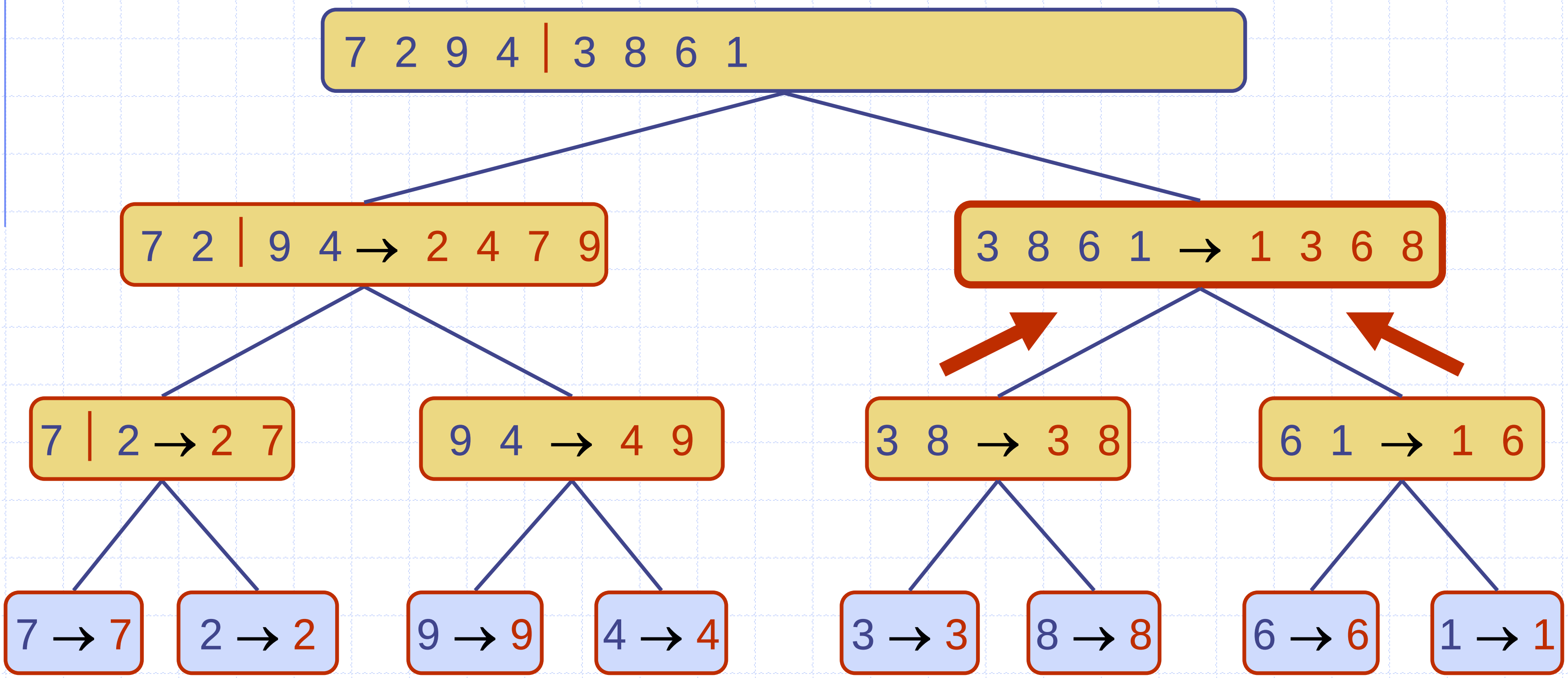
Execution Example (cont.)

- Merge



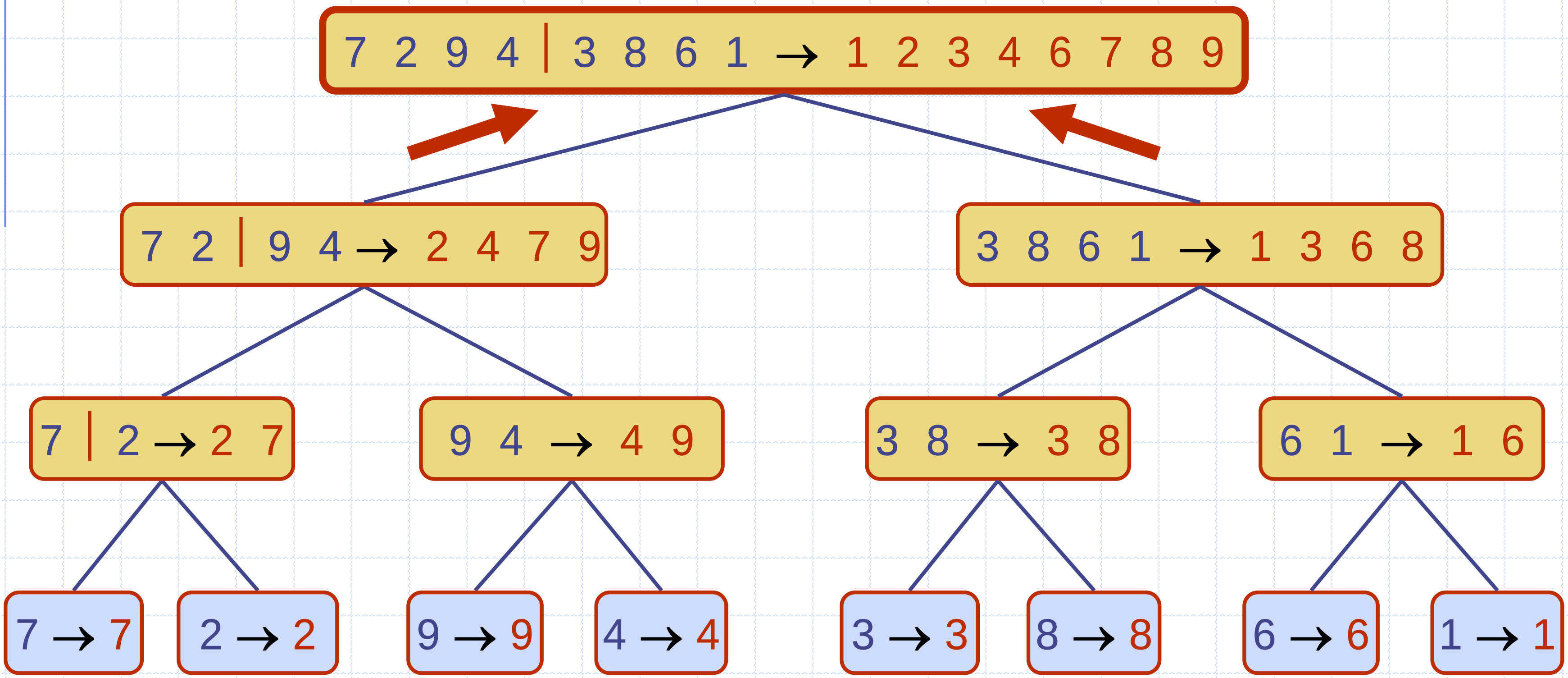
Execution Example (cont.)

- Recursive call, …, merge, merge



Execution Example (cont.)

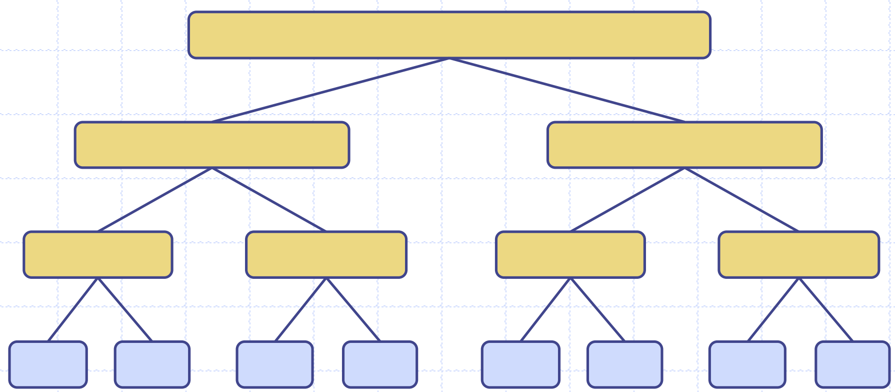
- Merge



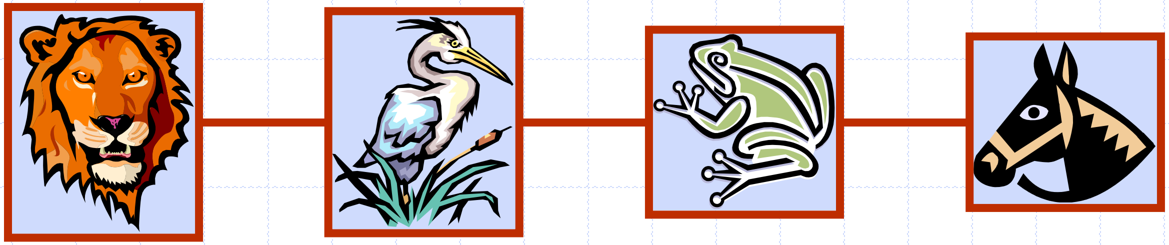
Analysis of Merge-Sort

- The height h of the merge-sort tree is $O(\log n)$
  - at each recursive call we divide in half the sequence,
- The overall amount or work done at the nodes of depth i is $O(n)$
  - we partition and merge 2^i sequences of size $n/2^i$
  - we make 2^{i+1} recursive calls
- Thus, the total running time of merge-sort is $O(n \log n)$

depth	#seqs	size
0	1	n
1	2	$n/2$
i	2^i	$n/2^i$
…	…	…

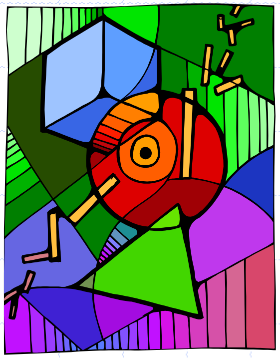


Sets



Set Operations

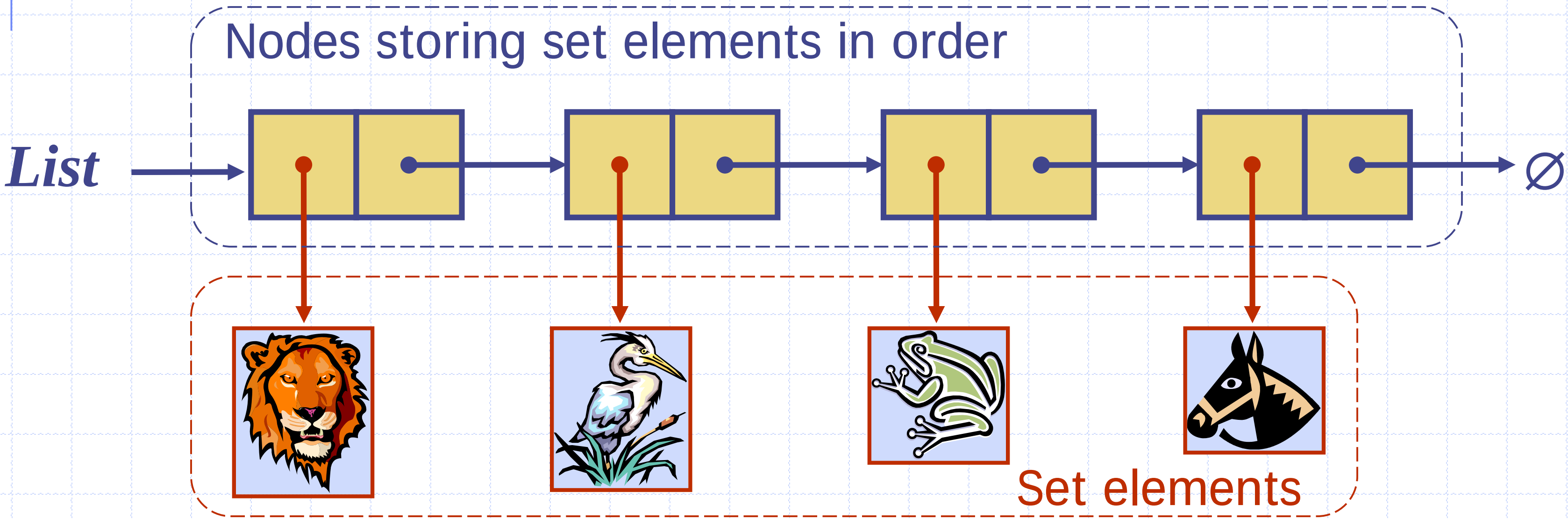
- We represent a set by the sorted sequence of its elements
- By specializing the auxliliary methods he generic merge algorithm can be used to perform basic set operations:
  - union
  - intersection
  - subtraction
- The running time of an operation on sets A and B should be at most $O(n_A + n_B)$



- Set union:
  - ***aIsLess***(a, S)
    $S.insertFirst(a)$
  - ***bIsLess***(b, S)
    $S.insertLast(b)$
  - ***bothAreEqual***(a, b, S)
    $S.\ insertLast(a)$
- Set intersection:
  - ***aIsLess***(a, S)
    { *do nothing* }
  - ***bIsLess***(b, S)
    { *do nothing* }
  - ***bothAreEqual***(a, b, S)
    $S.\ insertLast(a)$

Storing a Set in a List

- We can implement a set with a list
- Elements are stored sorted according to some canonical ordering
- The space used is $O(n)$



Generic Merging

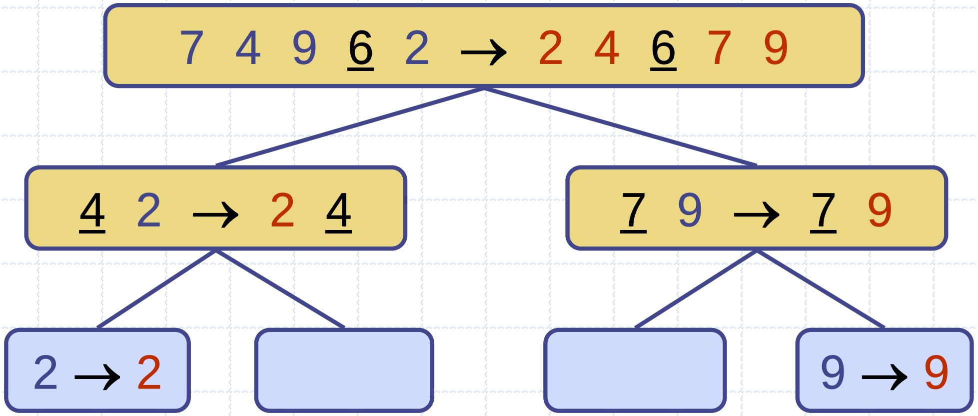
- Generalized merge of two sorted lists A and B
- Template method genericMerge
- Auxiliary methods
  - aIsLess
  - bIsLess
  - bothAreEqual
- Runs in $O(n_A + n_B)$ time provided the auxiliary methods run in $O(1)$ time

```
Algorithm genericMerge(A, B)
    S ← empty sequence
    while ¬A.isEmpty() ∧ ¬B.isEmpty()
        a ← A.first().element();  b ← B.first().element()
        if a < b
            aIsLess(a, S);  A.remove(A.first())
        else if b < a
            bIsLess(b, S);  B.remove(B.first())
        else { b = a }
            bothAreEqual(a, b, S)
            A.remove(A.first());  B.remove(B.first())
    while ¬A.isEmpty()
        aIsLess(a, S);  A.remove(A.first())
    while ¬B.isEmpty()
        bIsLess(b, S);  B.remove(B.first())
    return S
```

Using Generic Merge for Set Operations

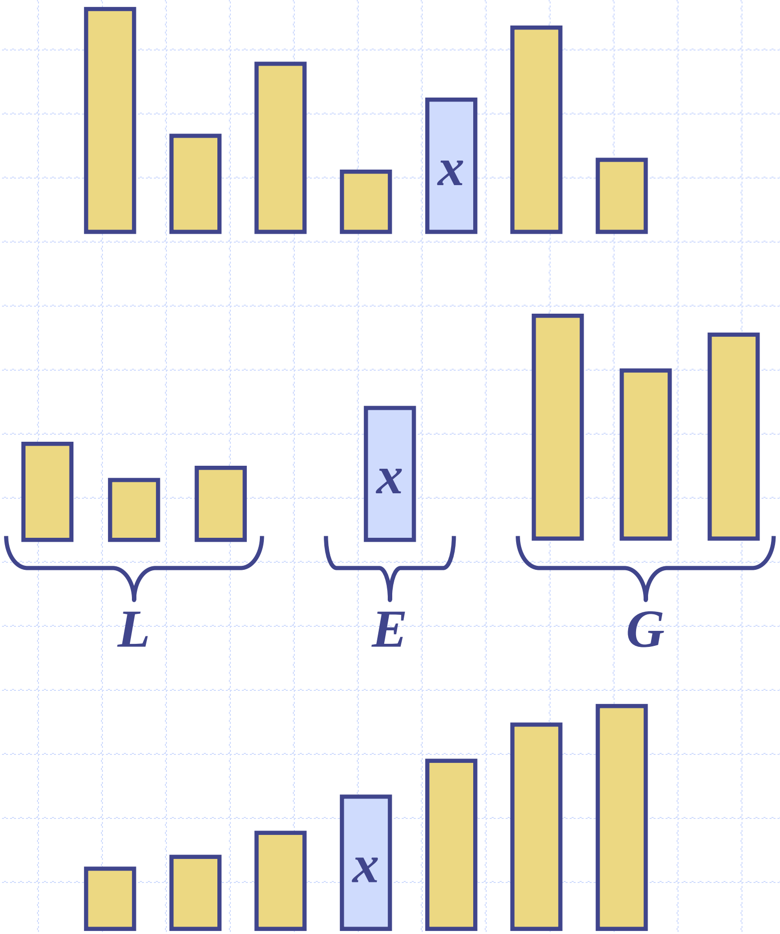
- Any of the set operations can be implemented using a generic merge
- For example:
  - For intersection: only copy elements that are duplicated in both list
  - For union: copy every element from both lists except for the duplicates
- All methods run in linear time.

Quick-Sort



Quick-Sort

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E and G



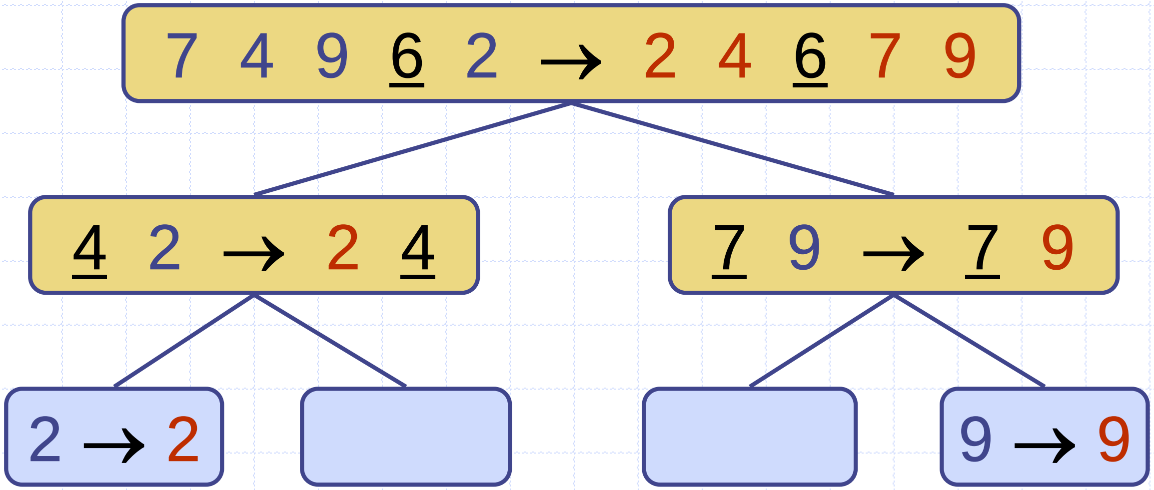
Partition

- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

```
Algorithm partition(S, p)
    Input sequence S, position p of pivot
    Output subsequences L, E, G of the
        elements of S less than, equal to,
        or greater than the pivot, resp.
    L, E, G ← empty sequences
    x ← S.remove(p)
    while ¬S.isEmpty()
        y ← S.remove(S.first())
        if y < x
            L.insertLast(y)
        else if y = x
            E.insertLast(y)
        else { y > x }
            G.insertLast(y)
    return L, E, G
```

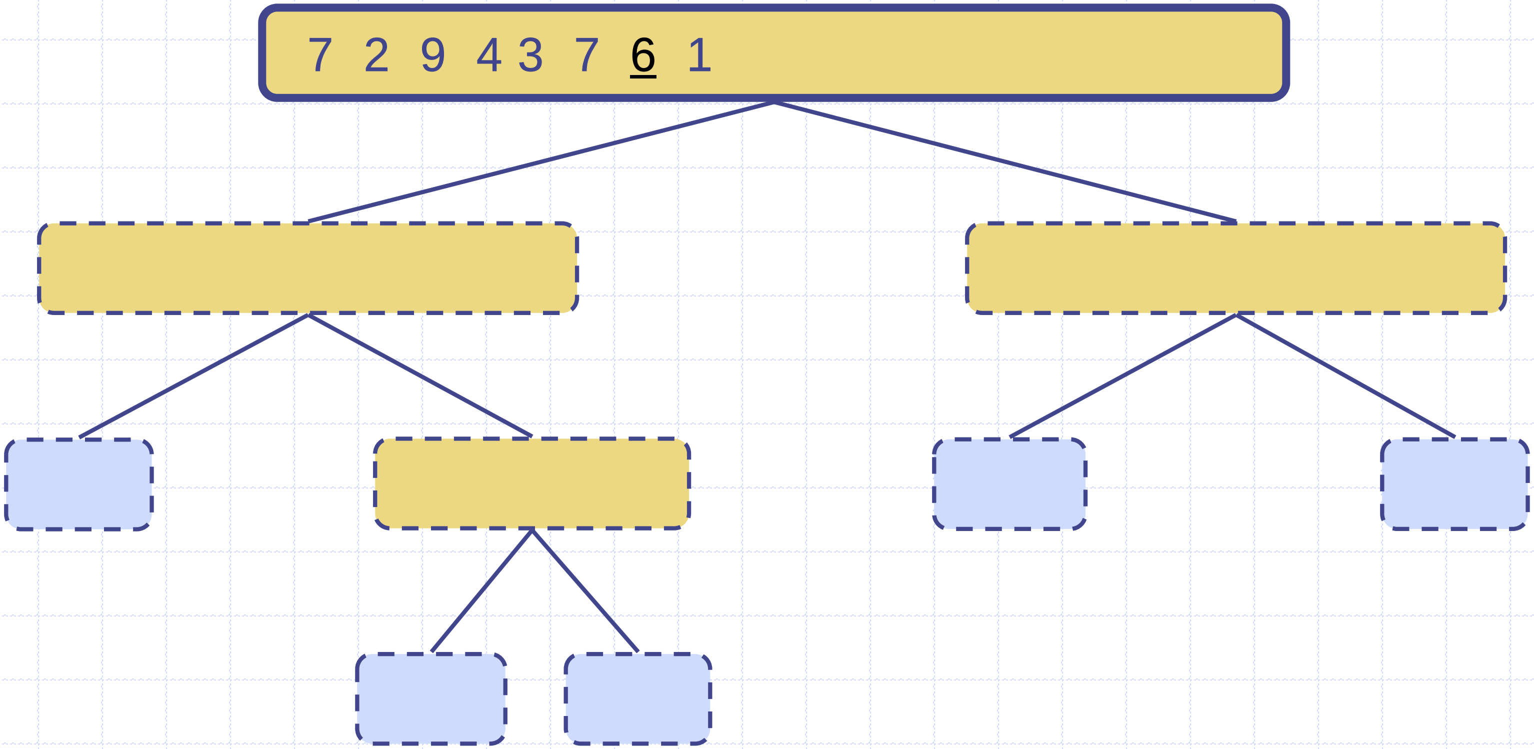
Quick-Sort Tree

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



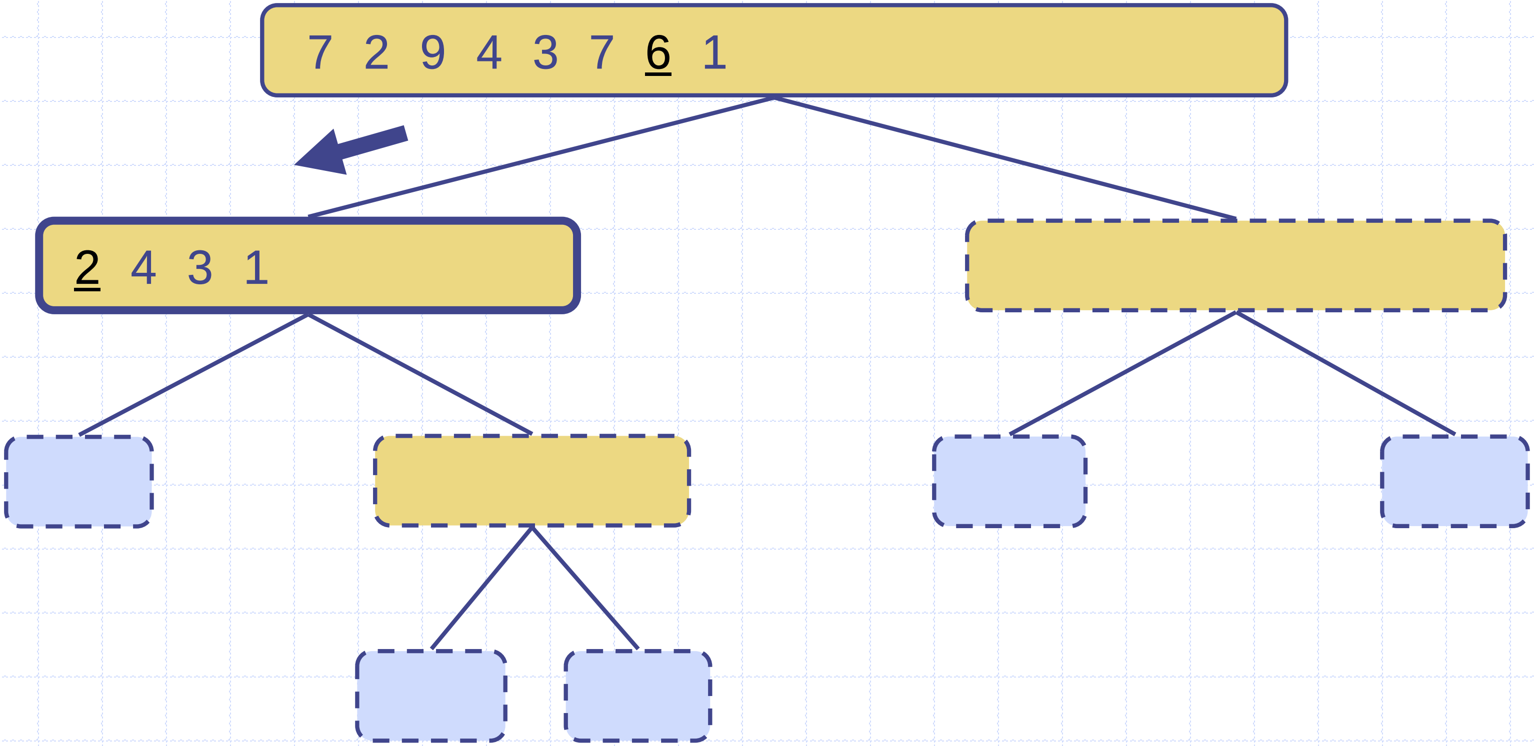
Execution Example

- Pivot selection



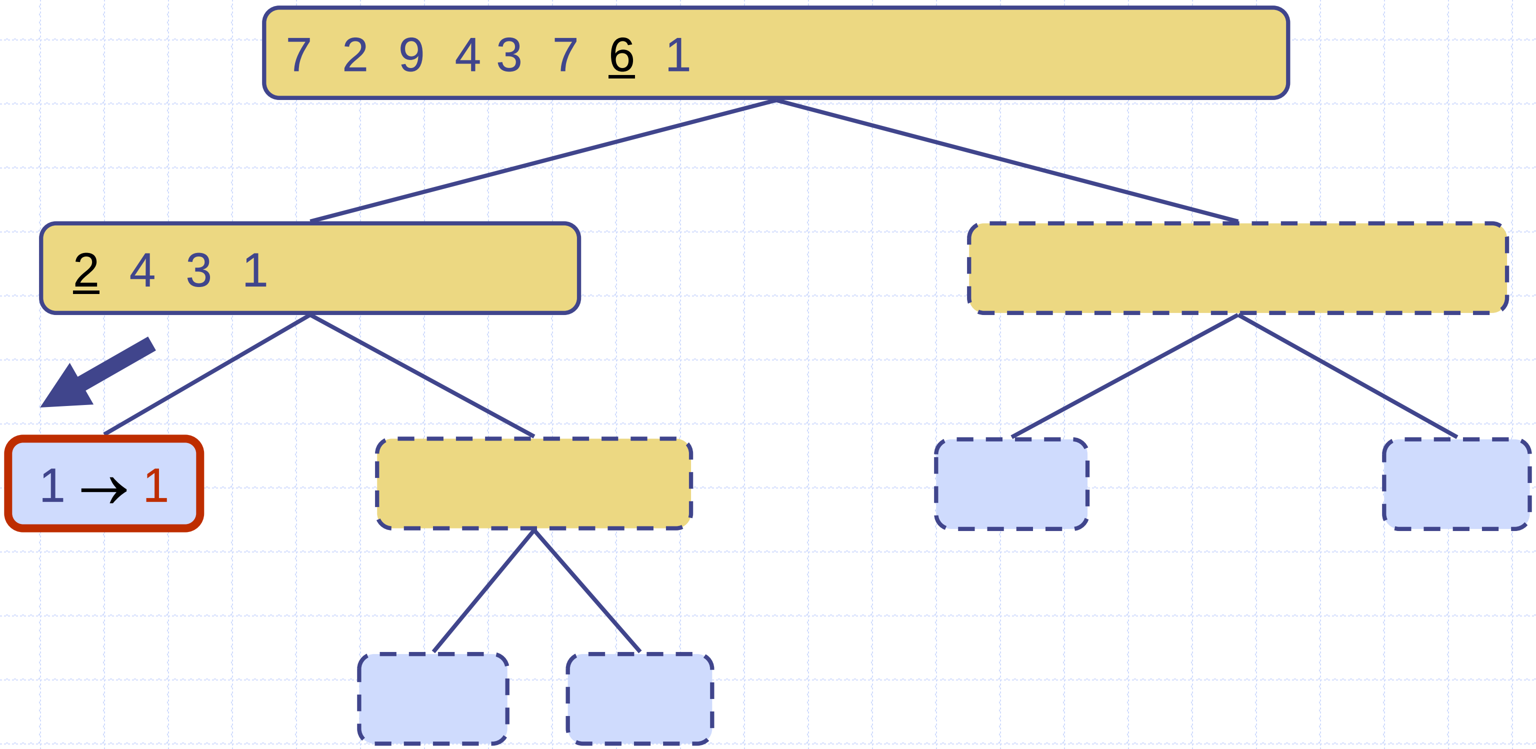
Execution Example (cont.)

- Partition, recursive call, pivot selection



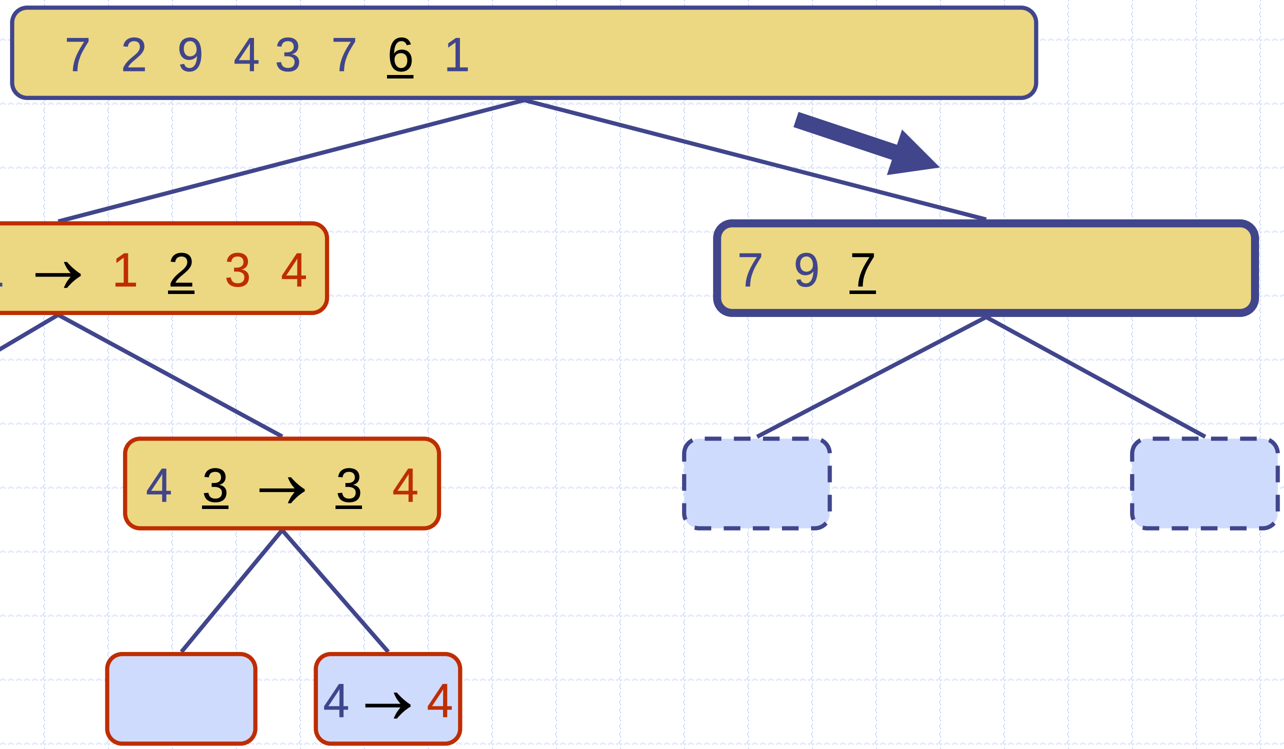
Execution Example (cont.)

- Partition, recursive call, base case



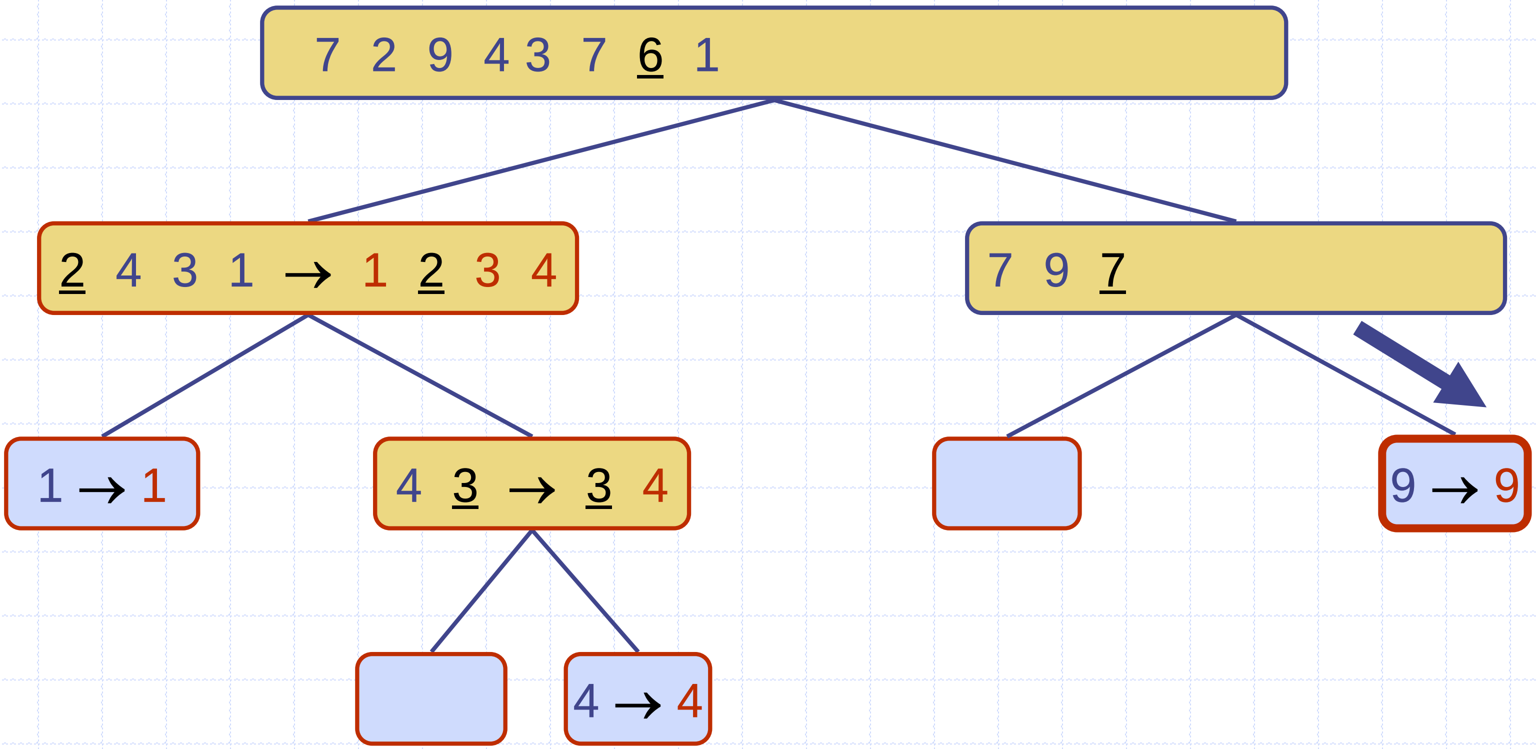
Execution Example (cont.)

- Recursive call, pivot selection



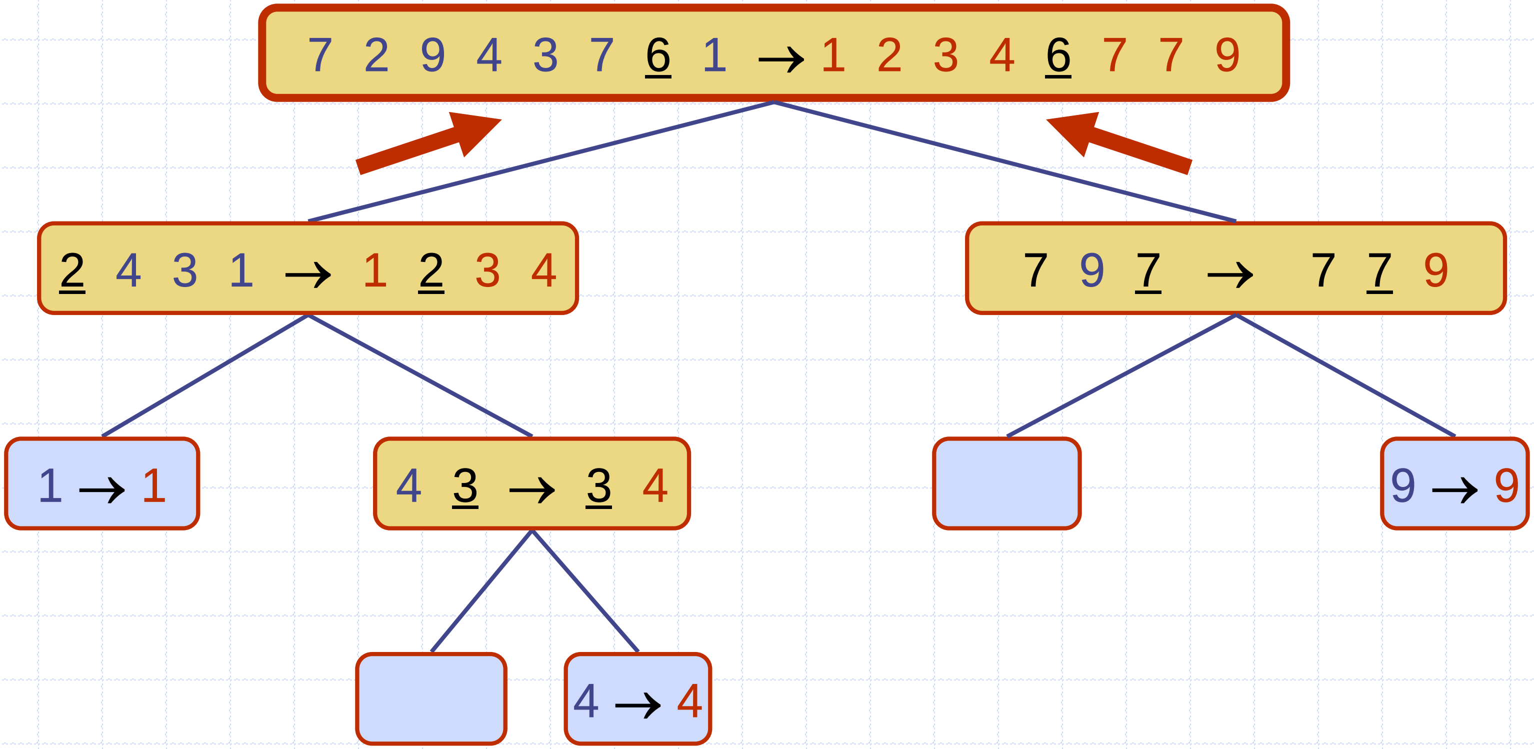
Execution Example (cont.)

- Partition, …, recursive call, base case



Execution Example (cont.)

- Join, join



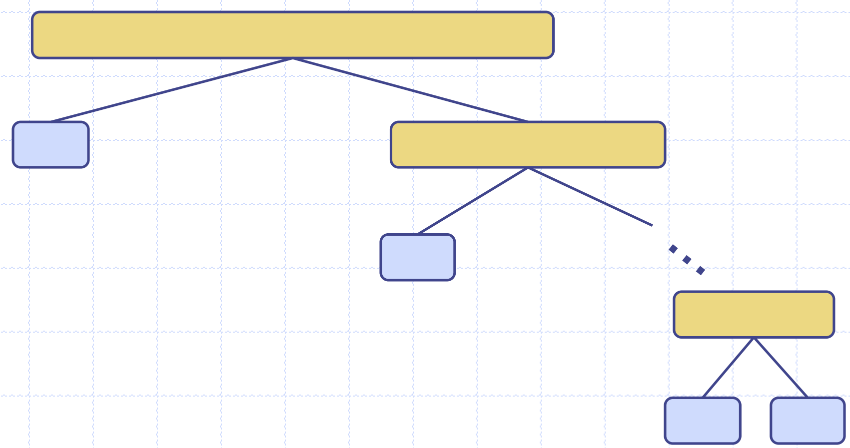
Worst-case Running Time

- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of L and G has size $n - 1$ and the other has size 0
- The running time is proportional to the sum

$$n + (n - 1) + \ldots + 2 + 1$$

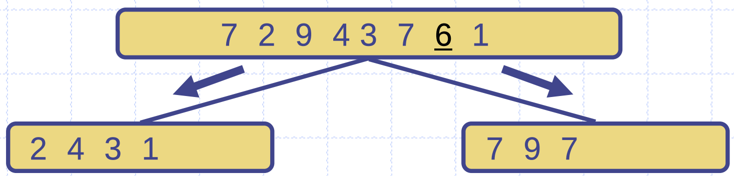
- Thus, the worst-case running time of quick-sort is $O(n^2)$

depth	time
0	n
1	$n - 1$
…	…
$n - 1$	1

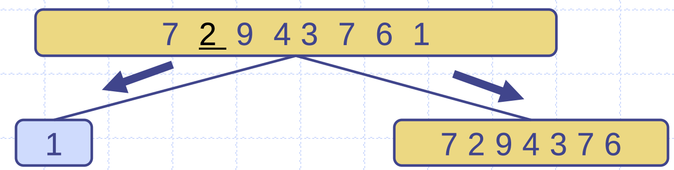


Expected Running Time

- Consider a recursive call of quick-sort on a sequence of size s
  - **Good call:** the sizes of L and G are each less than $3s/4$
  - **Bad call:** one of L and G has size greater than $3s/4$

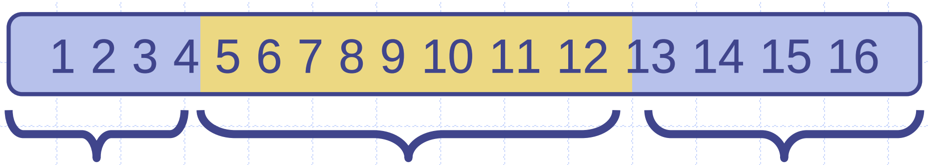


Good call



Bad call

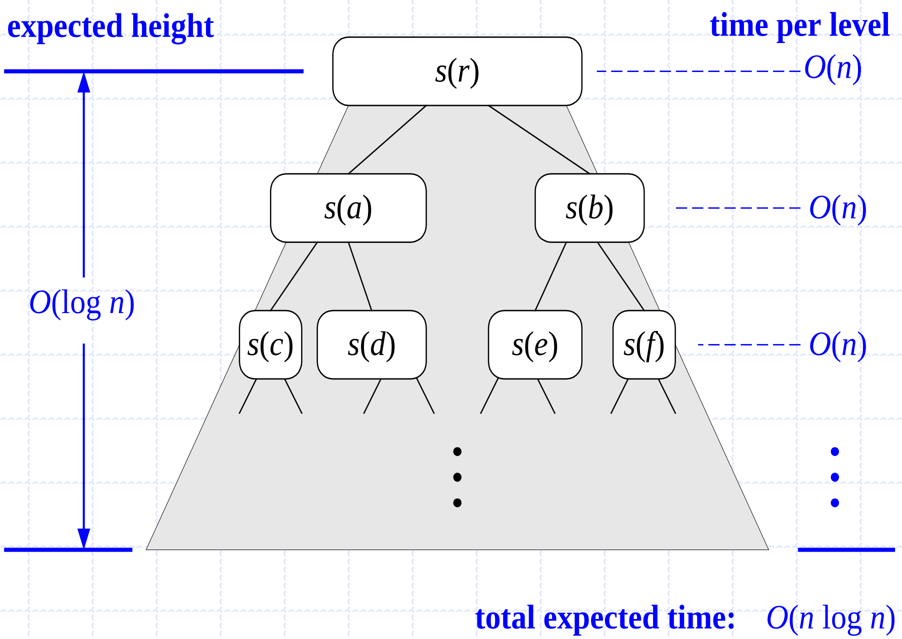
- A call is good with probability $1/2$
  - 1/2 of the possible pivots cause good calls:



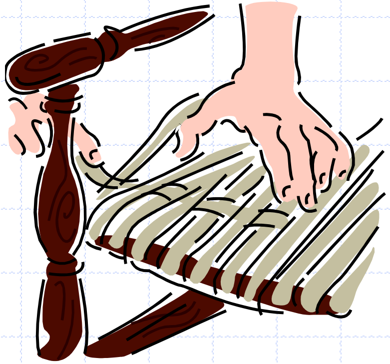
Bad pivots **Good pivots** **Bad pivots**

Expected Running Time, Part 2

- Probabilistic Fact: The expected number of coin tosses required in order to get $\boldsymbol{k}$ heads is $2\boldsymbol{k}$
- For a node of depth $\boldsymbol{i}$, we expect
  - $\boldsymbol{i}/2$ ancestors are good calls
  - The size of the input sequence for the current call is at most $(3/4)^{\boldsymbol{i}/2}\boldsymbol{n}$
- Therefore, we have
  - For a node of depth $2\log_{4/3}\boldsymbol{n}$, the expected input size is one
  - The expected height of the quick-sort tree is $\boldsymbol{O}(\log \boldsymbol{n})$
- The amount or work done at the nodes of the same depth is $\boldsymbol{O}(\boldsymbol{n})$
- Thus, the expected running time of quick-sort is $\boldsymbol{O}(\boldsymbol{n} \log \boldsymbol{n})$



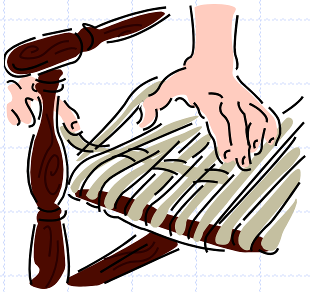
In-Place Quick-Sort



- Quick-sort can be implemented to run in-place
- In the partition step, we use replace operations to rearrange the elements of the input sequence such that
  - the elements less than the pivot have rank less than ***h***
  - the elements equal to the pivot have rank between ***h*** and ***k***
  - the elements greater than the pivot have rank greater than ***k***
- The recursive calls consider
  - elements with rank less than ***h***
  - elements with rank greater than ***k***

Algorithm ***inPlaceQuickSort***(***S, l, r***)
  Input sequence ***S***, ranks ***l*** and ***r***
  Output sequence ***S*** with the
    elements of rank between ***l*** and ***r***
    rearranged in increasing order
  if $l \geq r$
    return
  $i \leftarrow$ a random integer between ***l*** and ***r***
  $x \leftarrow$ ***S.elemAtRank***(***i***)
  $(h, k) \leftarrow$ ***inPlacePartition***(***x***)
  inPlaceQuickSort(***S, l, h*** − 1)
  inPlaceQuickSort(***S, k*** + 1, ***r***)

In-Place Partitioning



- Perform the partition using two indices to split S into L and E U G (a similar method can split E U G into E and G).

j k

3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **<u>6</u>** 9 (pivot = 6)

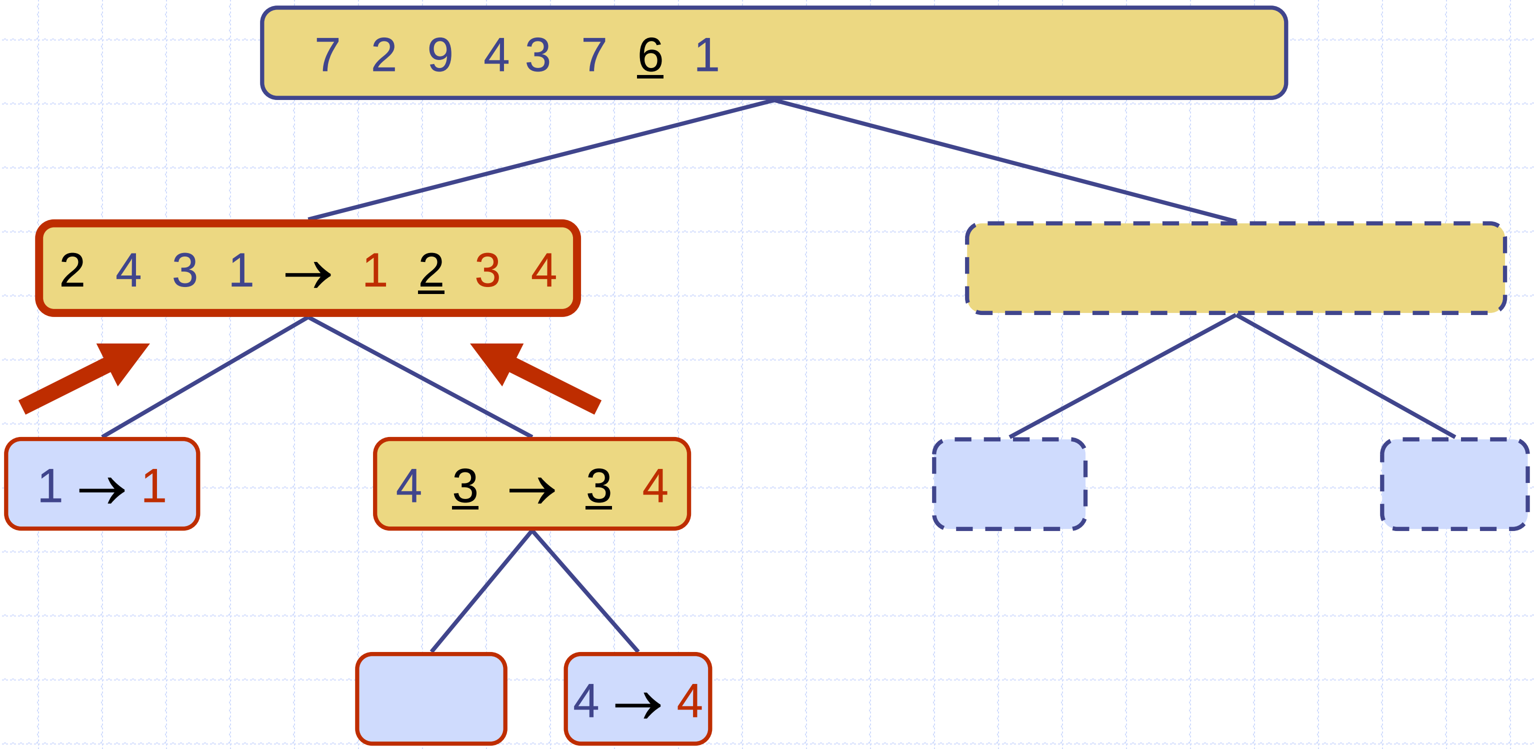
- Repeat until j and k cross:
  - Scan j to the right until finding an element $\geq$ x.
  - Scan k to the left until finding an element < x.
  - Swap elements at indices j and k

j k

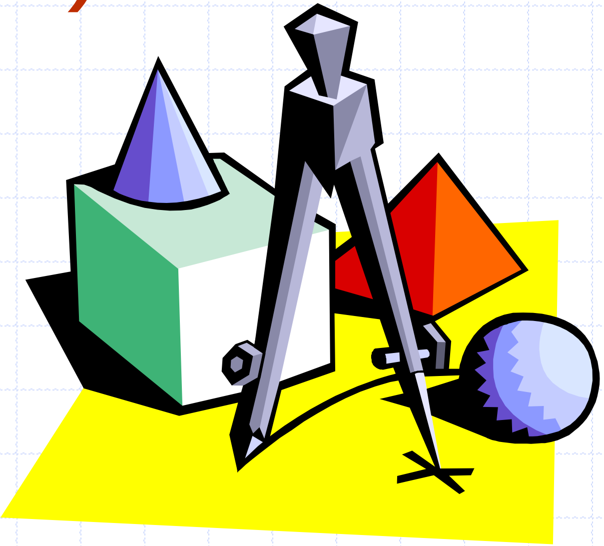
3 2 5 1 0 $\boxed{7}$ 3 5 9 $\boxed{2}$ 7 9 8 9 7 **<u>6</u>** 9

Execution Example (cont.)

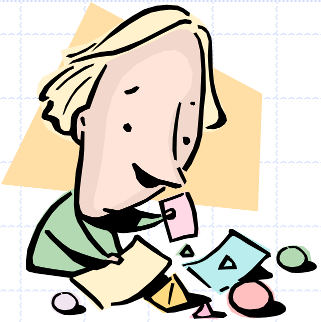
- Recursive call, ..., base case, join



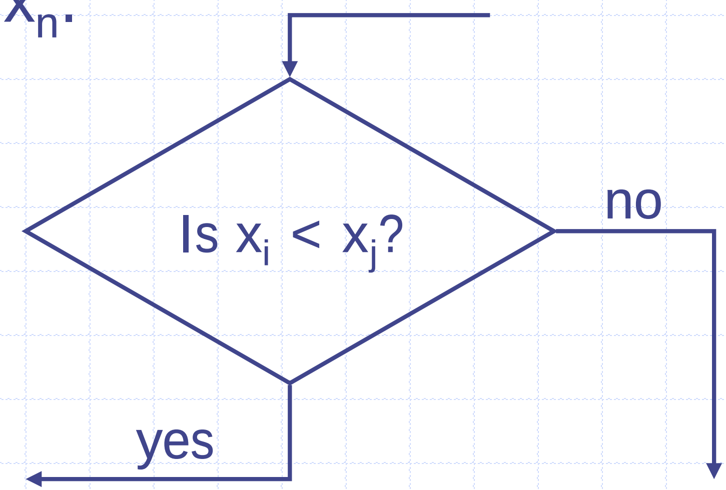
A Lower Bound on Comparison-based Sorting (§4.4)



Comparison-Based Sorting

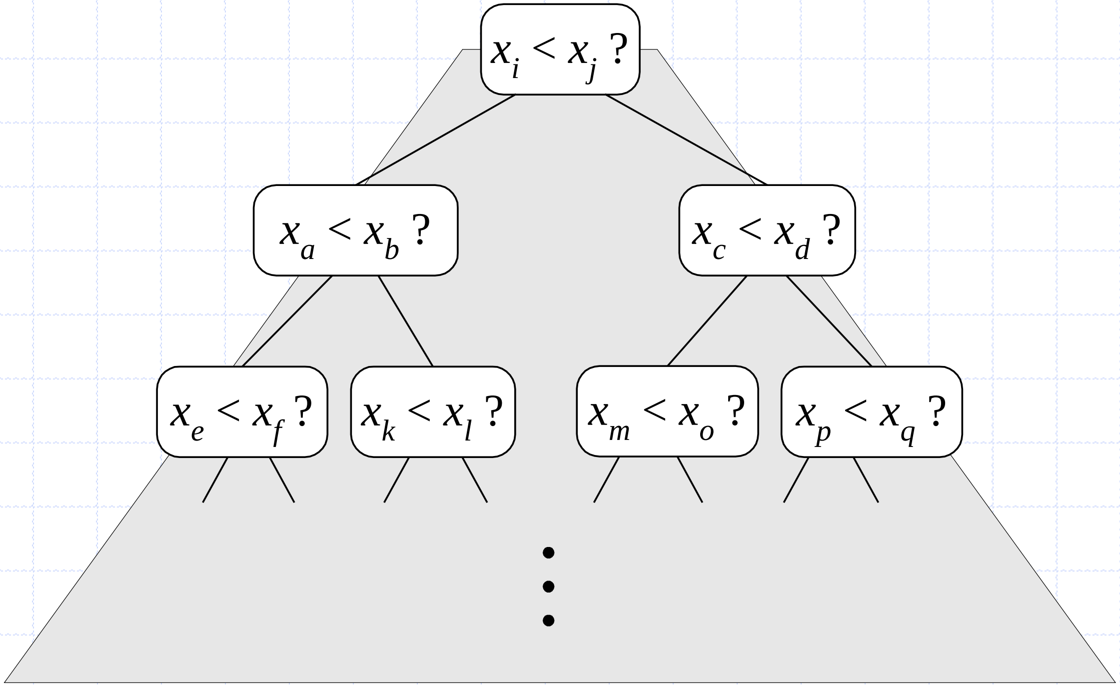


- Many sorting algorithms are comparison based.
  - They sort by making comparisons between pairs of objects
  - Examples: bubble-sort, selection-sort, insertion-sort, heap-sort, merge-sort, quick-sort, ...
- Let us therefore derive a lower bound on the running time of any algorithm that uses comparisons to sort n elements, $x_1, x_2, \ldots, x_n$.



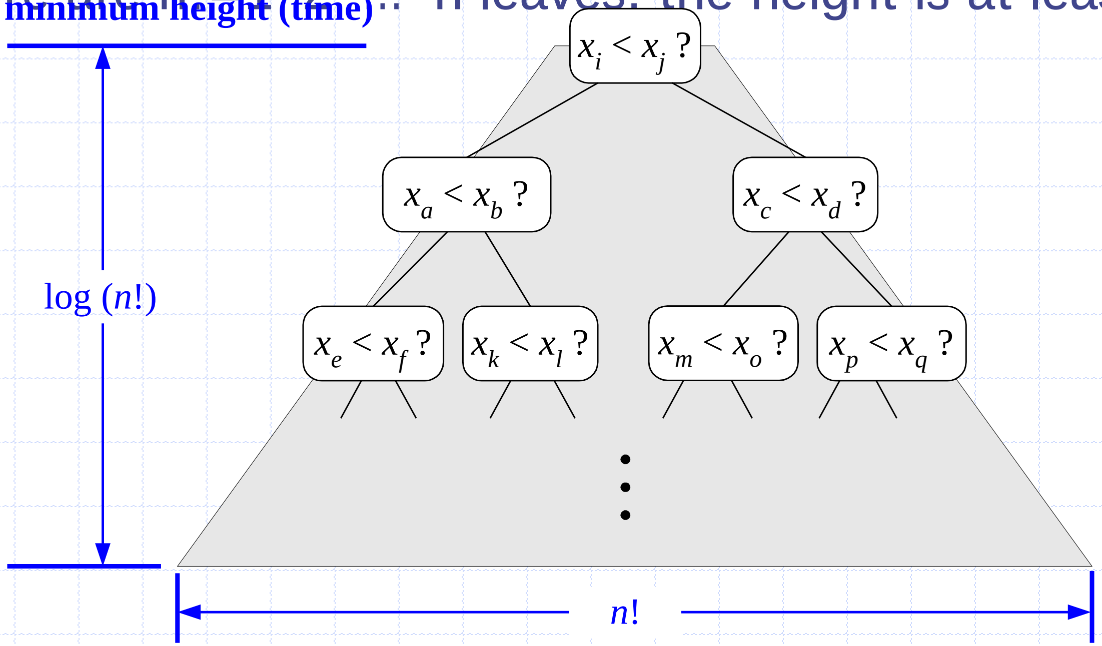
Counting Comparisons

- Let us just count comparisons then.
- Each possible run of the algorithm corresponds to a root-to-leaf path in a **decision tree**

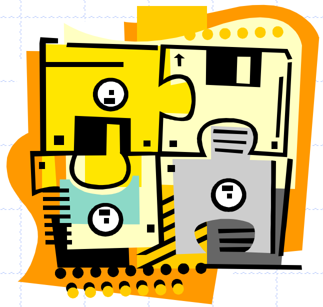


Decision Tree Height

- The height of this decision tree is a lower bound on the running time
- Every possible input permutation must lead to a separate leaf output.
  - If not, some input ...4...5... would have same output ordering as ...5...4..., which would be wrong.
- Since there are n!=1*2*...*n leaves, the height is at least log (n!)



The Lower Bound

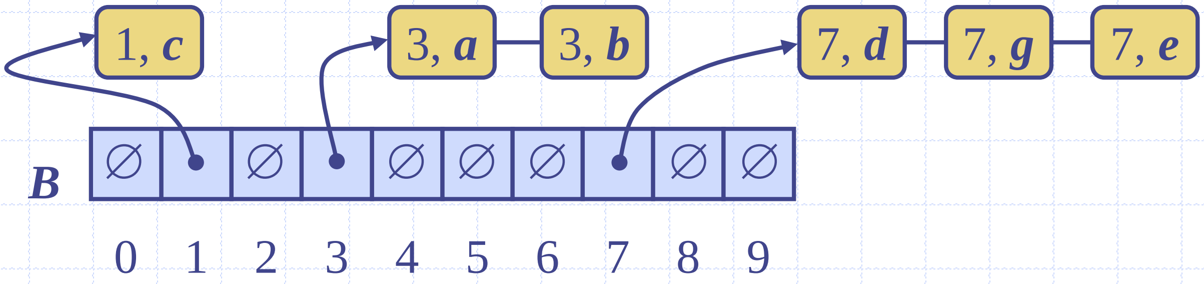


- Any comparison-based sorting algorithms takes at least log (n!) time
- Therefore, any such algorithm takes time at least

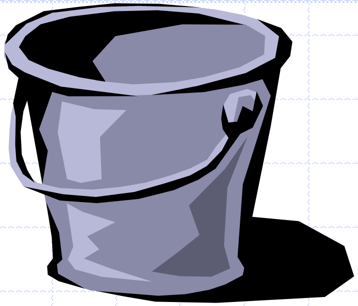
$$\log (n!) \geq \log \left(\frac{n}{2}\right)^{\frac{n}{2}} = (n/2)\log (n/2).$$

- That is, any comparison-based sorting algorithm must run in Ω(n log n) time.

Bucket-Sort and Radix-Sort



Bucket-Sort (§ 4.5.1)



- Let be S be a sequence of n (key, element) items with keys in the range $[0, N-1]$
- Bucket-sort uses the keys as indices into an auxiliary array B of sequences (buckets)

  Phase 1: Empty sequence S by moving each item (k, o) into its bucket $B[k]$

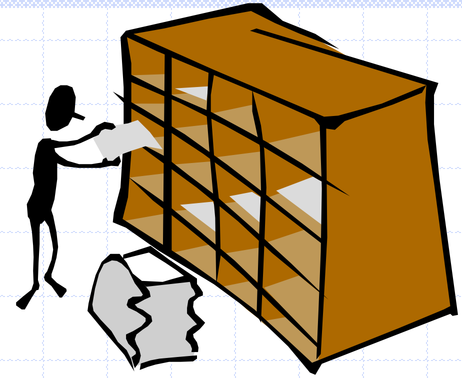
  Phase 2: For $i = 0, \ldots, N-1$, move the items of bucket $B[i]$ to the end of sequence S
- Analysis:
  - Phase 1 takes $O(n)$ time
  - Phase 2 takes $O(n + N)$ time

  Bucket-sort takes $O(n + N)$ time

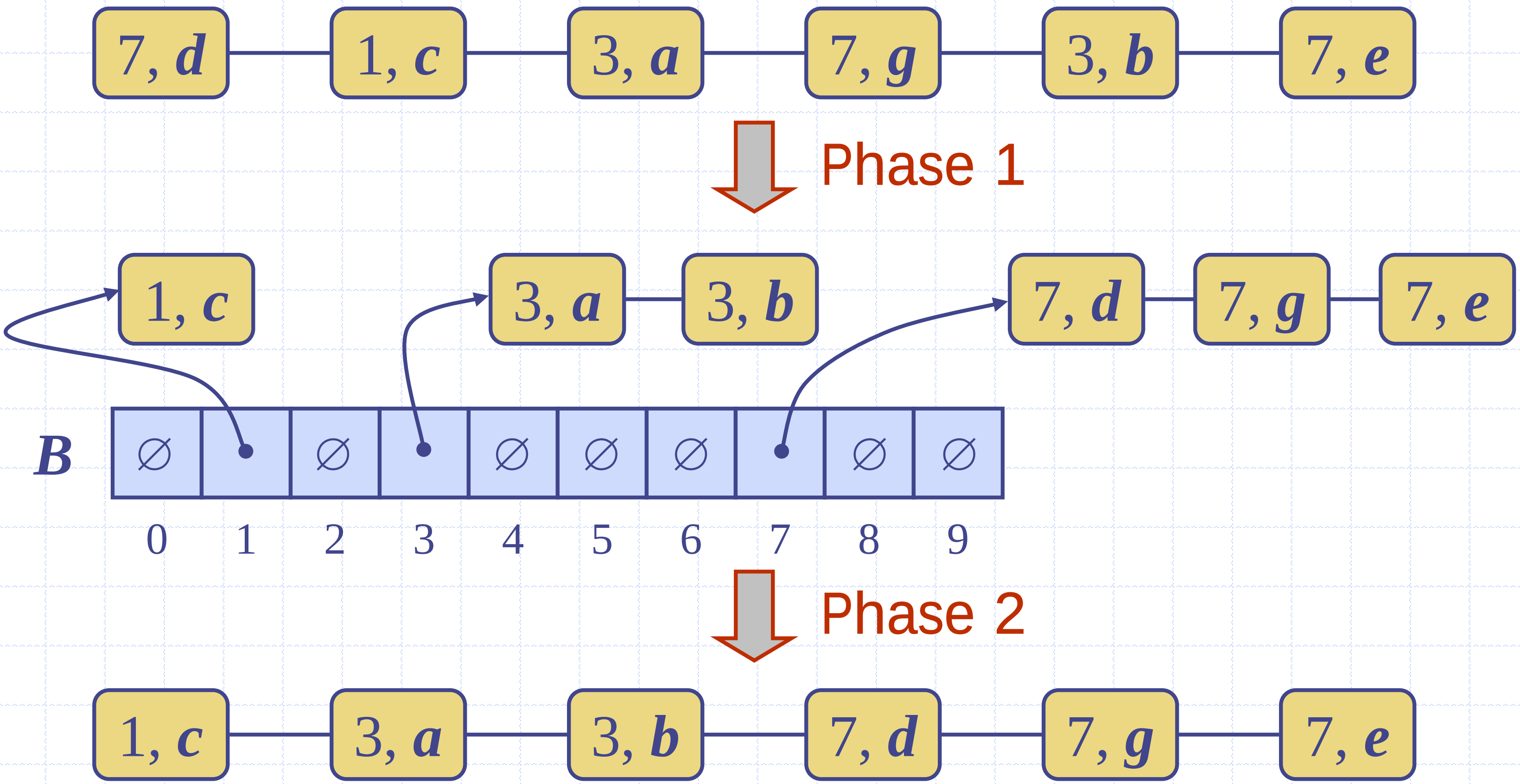
Algorithm *bucketSort*(S, N)
    Input sequence S of (key, element) items with keys in the range $[0, N-1]$
    Output sequence S sorted by increasing keys

```
B ← array of N empty sequences
while ¬S.isEmpty()
    f ← S.first()
    (k, o) ← S.remove(f)
    B[k].insertLast((k, o))
for i ← 0 to N – 1
    while ¬B[i].isEmpty()
        f ← B[i].first()
        (k, o) ← B[i].remove(f)
        S.insertLast((k, o))
```

Example



- Key range [0, 9]



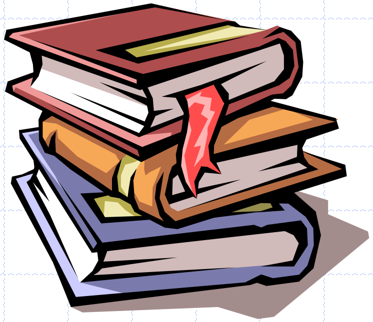
Properties and Extensions

- **Key-type Property**
  - The keys are used as indices into an array and cannot be arbitrary objects
  - No external comparator
- **Stable Sort Property**
  - The relative order of any two items with the same key is preserved after the execution of the algorithm

Extensions

- Integer keys in the range $[a, b]$
  - Put item (k, o) into bucket $B[k - a]$
- String keys from a set D of possible strings, where D has constant size (e.g., names of the 50 U.S. states)
  - Sort D and compute the rank $r(k)$ of each string k of D in the sorted sequence
  - Put item (k, o) into bucket $B[r(k)]$

Lexicographic Order



- A d-tuple is a sequence of d keys $(k_1, k_2, \ldots, k_d)$, where key k_i is said to be the i-th dimension of the tuple
- Example:
  - The Cartesian coordinates of a point in space are a 3-tuple
- The lexicographic order of two d-tuples is recursively defined as follows

$$(x_1, x_2, \ldots, x_d) < (y_1, y_2, \ldots, y_d)$$

$$\Leftrightarrow$$

$$x_1 < y_1 \lor x_1 = y_1 \land (x_2, \ldots, x_d) < (y_2, \ldots, y_d)$$

I.e., the tuples are compared by the first dimension, then by the second dimension, etc.

Lexicographic-Sort

- Let C_i be the comparator that compares two tuples by their i-th dimension
- Let $stableSort(S, C)$ be a stable sorting algorithm that uses comparator C
- Lexicographic-sort sorts a sequence of d-tuples in lexicographic order by executing d times algorithm *stableSort*, one per dimension
- Lexicographic-sort runs in $O(dT(n))$ time, where $T(n)$ is the running time of *stableSort*

Algorithm *lexicographicSort*(S)
  Input sequence S of d-tuples
  Output sequence S sorted in lexicographic order

  for $i \leftarrow d$ **downto** 1
    $stableSort(S, C_i)$

Example:

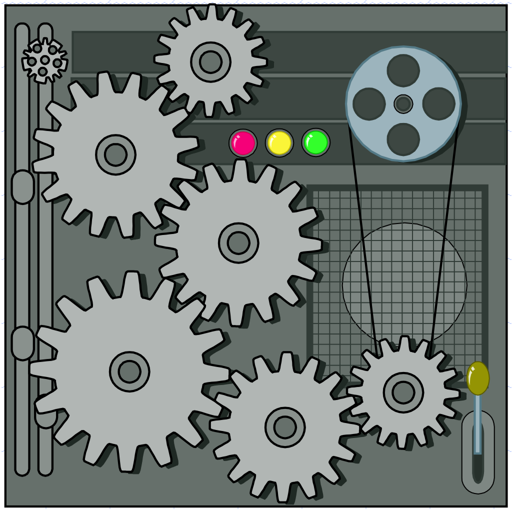
(7,4,6) (5,1,5) (2,4,6) (2, 1, 4) (3, 2, 4)

(2, 1, 4) (3, 2, 4) (5,1,5) (7,4,6) (2,4,6)

(2, 1, 4) (5,1,5) (3, 2, 4) (7,4,6) (2,4,6)

(2, 1, 4) (2,4,6) (3, 2, 4) (5,1,5) (7,4,6)

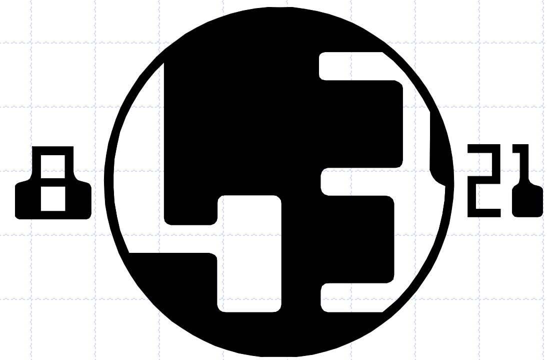
Radix-Sort (§ 4.5.2)



- Radix-sort is a specialization of lexicographic-sort that uses bucket-sort as the stable sorting algorithm in each dimension
- Radix-sort is applicable to tuples where the keys in each dimension i are integers in the range $[0, N-1]$
- Radix-sort runs in time $O(d(n+N))$

Algorithm *radixSort*(S, N)
  Input sequence S of d-tuples such that $(0, \ldots, 0) \le (x_1, \ldots, x_d)$ and $(x_1, \ldots, x_d) \le (N-1, \ldots, N-1)$ for each tuple $(x_1, \ldots, x_d)$ in S
  Output sequence S sorted in lexicographic order
  for $i \leftarrow d$ **downto** 1
    bucketSort(S, N)

Radix-Sort for Binary Numbers



- Consider a sequence of n b-bit integers

  $$x = x_{b-1} \dots x_1 x_0$$

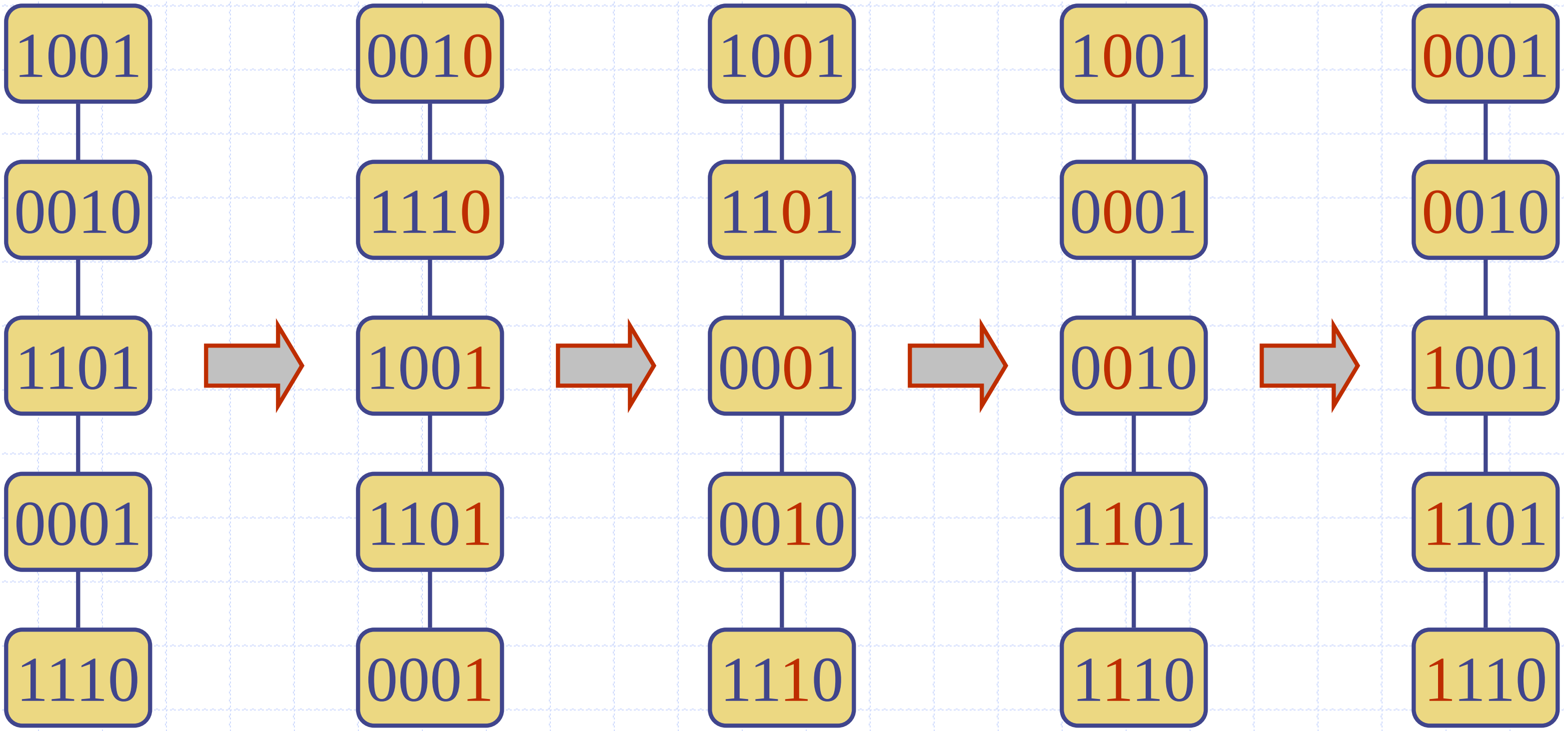
- We represent each element as a b-tuple of integers in the range $[0, 1]$ and apply radix-sort with $N = 2$
- This application of the radix-sort algorithm runs in $O(bn)$ time
- For example, we can sort a sequence of 32-bit integers in linear time

Algorithm ***binaryRadixSort***(S)

  Input sequence S of b-bit integers
  Output sequence S sorted
  replace each element x of S with the item $(0, x)$
  for $i \leftarrow 0$ **to** $b - 1$
    replace the key k of each item (k, x) of S with bit x_i of x
    bucketSort$(S, 2)$

Example

- Sorting a sequence of 4-bit integers



Summary of Sorting Algorithms (§4.6)

Algorithm	Time	Notes
selection-sort	$O(n^2)$	◈ slow ◈ in-place ◈ for small data sets (< 1K)
insertion-sort	$O(n^2)$	◈ slow ◈ in-place ◈ for small data sets (< 1K)
heap-sort	$O(n \log n)$	◈ fast ◈ in-place ◈ for large data sets (1K — 1M)
merge-sort	$O(n \log n)$	◈ fast ◈ sequential data access ◈ for huge data sets (> 1M)

Selection (§4.7)

The Selection Problem

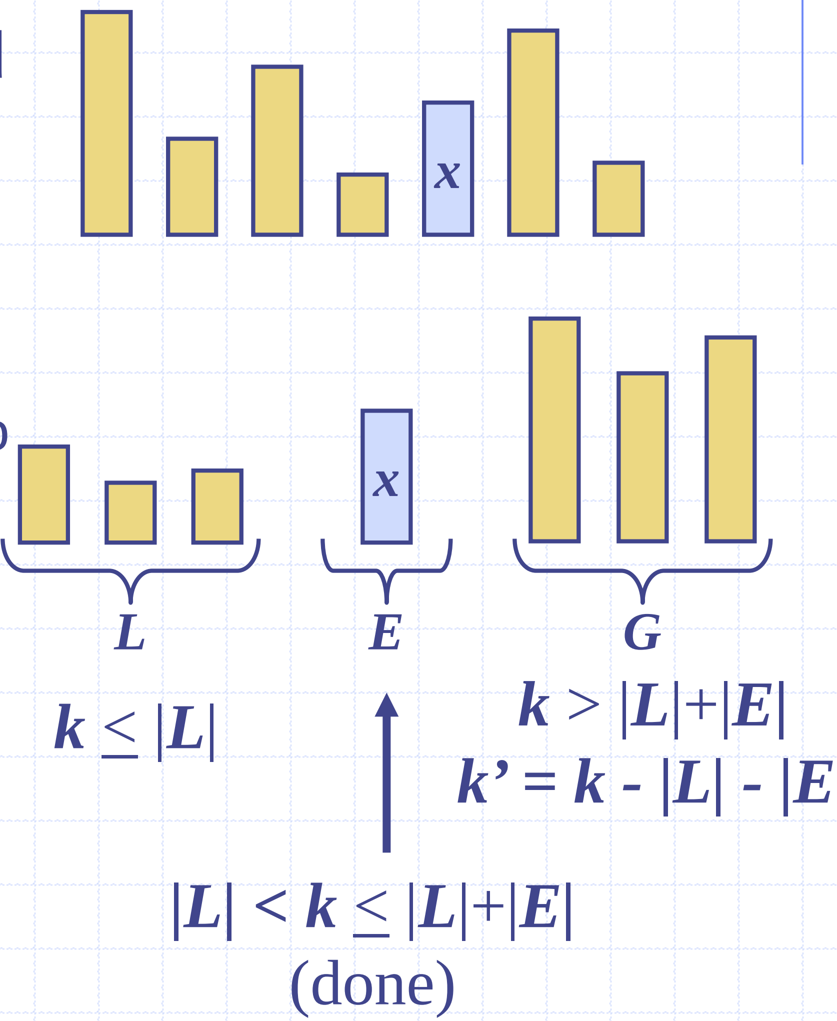
- Given an integer k and n elements $x_1, x_2, \ldots, x_n$, taken from a total order, find the k-th smallest element in this set.
- Of course, we can sort the set in O(n log n) time and then index the k-th element.

k=3   7 4 9 <u>6</u> 2 → 2 4 <u>6</u> 7 9

- Can we solve the selection problem faster?

Quick-Select

- Quick-select is a **randomized** selection algorithm based on the prune-and-search paradigm:
  - Prune: pick a random element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Search: depending on k, either answer is in E, or we need to recurse in either L or G



Partition

- We partition an input sequence as in the quick-sort algorithm:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-select takes $O(n)$ time

Algorithm *partition*(S, p)

Input sequence S, position p of pivot

Output subsequences L, E, G of the elements of S less than, equal to, or greater than the pivot, resp.

$L, E, G \leftarrow$ empty sequences

$x \leftarrow S.remove(p)$

while $\neg S.isEmpty()$

  $y \leftarrow S.remove(S.first())$

  if $y < x$

    $L.insertLast(y)$

  else if $y = x$

    $E.insertLast(y)$

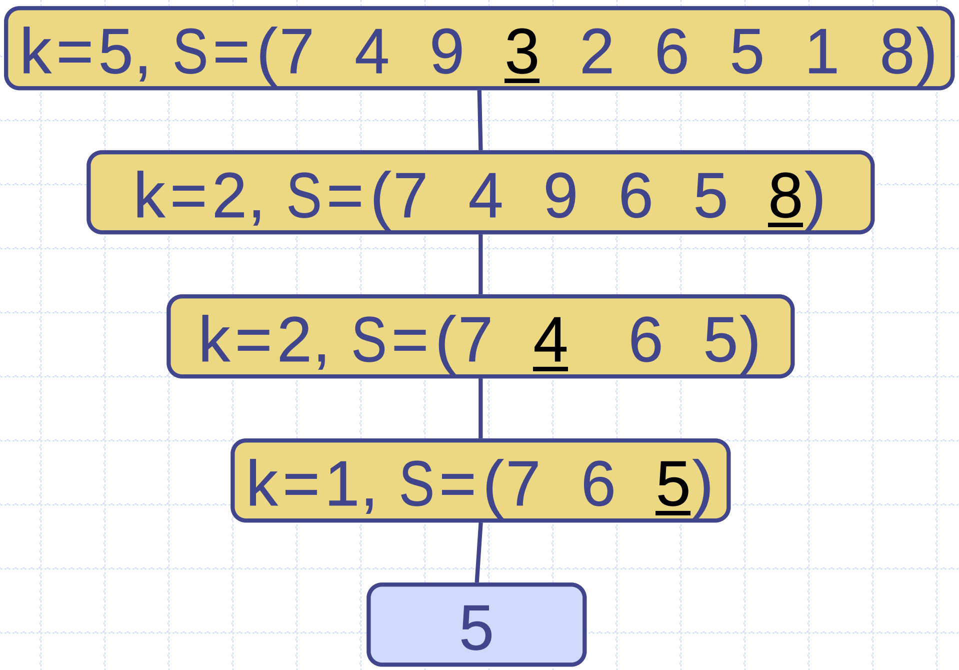
  else $\{ y > x \}$

    $G.insertLast(y)$

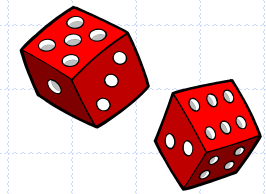
return L, E, G

Quick-Select Visualization

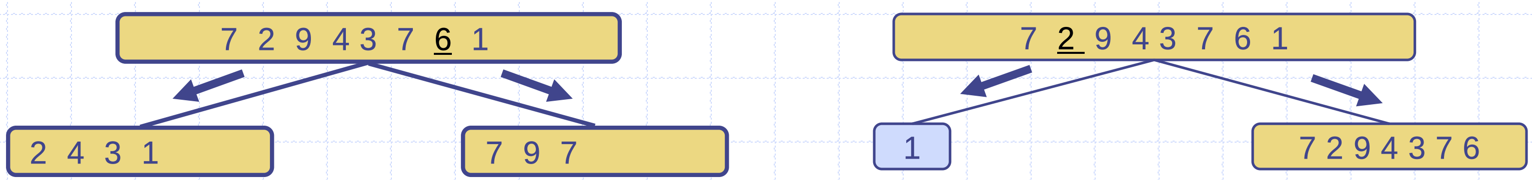
- An execution of quick-select can be visualized by a recursion path
  - Each node represents a recursive call of quick-select, and stores k and the remaining sequence



Expected Running Time

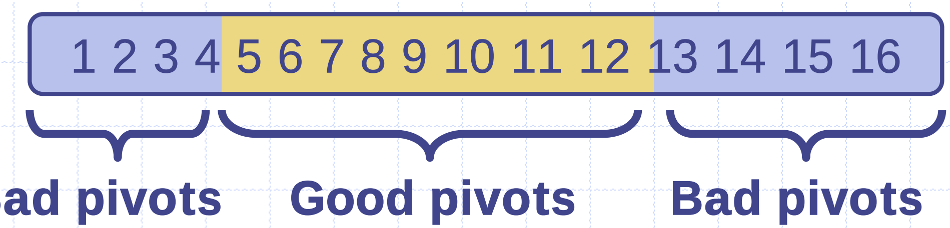


- Consider a recursive call of quick-select on a sequence of size s
  - **Good call:** the sizes of L and G are each less than $3s/4$
  - **Bad call:** one of L and G has size greater than $3s/4$

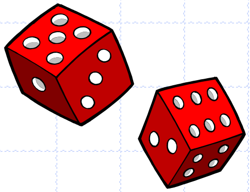


Good call **Bad call**

- A call is good with probability $1/2$
  - 1/2 of the possible pivots cause good calls:



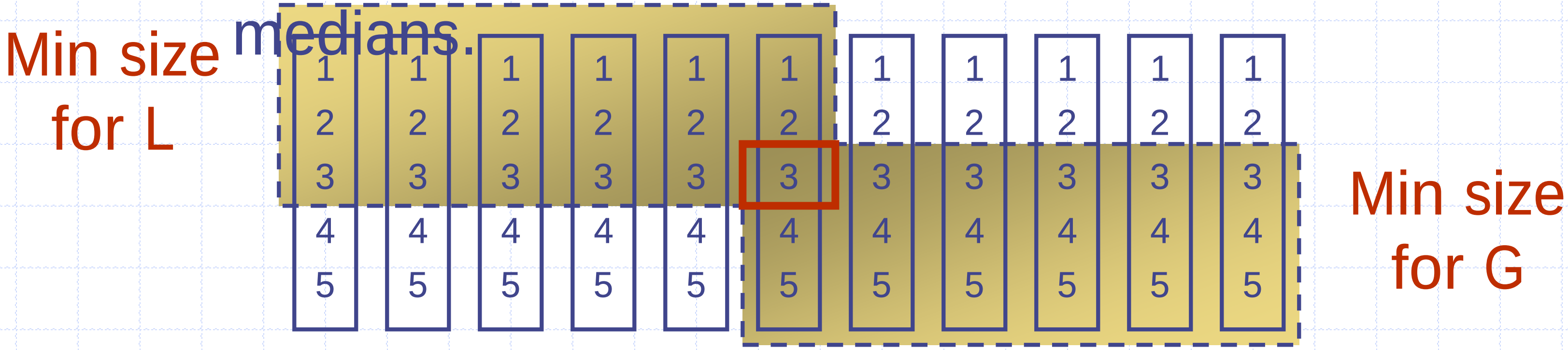
Expected Running Time, Part 2



- Probabilistic Fact #1: The expected number of coin tosses required in order to get one head is two
- Probabilistic Fact #2: Expectation is a linear function:
  - $E(X + Y) = E(X) + E(Y)$
  - $E(cX) = cE(X)$
- Let T(n) denote the expected running time of quick-select.
- By Fact #2,
  - $T(n) \leq T(3n/4) + bn*(\text{expected \# of calls before a good call})$
- By Fact #1,
  - $T(n) \leq T(3n/4) + 2bn$
- That is, T(n) is a geometric series:
  - $T(n) \leq 2bn + 2b(3/4)n + 2b(3/4)^2 n + 2b(3/4)^3 n + \ldots$
- So T(n) is O(n).
- We can solve the selection problem in O(n) expected time.

Deterministic Selection

- We can do selection in O(n) worst-case time.
- Main idea: recursively use the selection algorithm itself to find a good pivot for quick-select:
  - Divide S into n/5 sets of 5 each
  - Find a median in each set
  - Recursively find the median of the “baby” medians.



- See Exercise C-4.24 for details of analysis.